Geophysics:
Figure of Earth

O'BRIEN'S

MATHEMATICAL TRACTS,

PART I.

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MATHEMATICAL TRACTS.

PART I.

MATHEMATICAL TRACTS,

PART I.

ON

LAPLACE'S COEFFICIENTS,

THE FIGURE OF THE EARTH,

THE MOTION OF A RIGID BODY ABOUT ITS CENTER OF GRAVITY,

AND

PRECESSION AND NUTATION.

BY

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PREFACE.

THE subjects treated of in the following Tracts are, Laplace's Coefficients; the Investigation of the Figure of the Earth on the Hypothesis of its Original Fluidity; the Equations of Motion of a Rigid Body about its Center of Gravity; and the Application of these Equations to the case of the Earth. The first of these subjects should be familiar to every Mathematical Student, both for its own sake, and also on account of the many branches of Physical Science to which it is applicable. The second subject is extremely interesting as a physical theory, bearing upon the original state of the Earth and of the planetary bodies; it is also well worthy of attention on account of the important and extensive observations which have been made in order to verify it. The Author has put both these subjects together, commencing with the Figure of the Earth, and introducing Laplace's Coefficients when occasion required them; this being perhaps the best

and simplest way of exhibiting the nature and use of these coefficients.

The Author has treated some parts of these subjects differently from the manner in which they are usually treated, and he hopes that by so doing he has avoided some intricate reasoning and troublesome calculation, and made the whole more accessible to students of moderate mathematical attainments than it has hitherto been.

In calculating the attractions of the Earth on any particle, he has arrived at the correct results, without considering diverging series as inadmissible; and this he conceives to be important, because there is evidently no good reason why a diverging series should not be as good a *symbolical* representative of a quantity as a converging series; or why there should be any occasion to enquire whether a series is diverging or converging, as long as we do not want to calculate its arithmetical value or determine its sign. Instances, it is true, have been brought forward by Poisson* in which the use of diverging series appears to lead to error; but if the reasoning employed in Chapter III. of these Tracts be not incorrect, this error is due to quite a different cause;

* See Bowditch's Laplace. Vol. II. p. 167.

as will be immediately perceived on referring to Articles 33, 34, 35, and 37.

The Author has deduced the equations of motion of a rigid body about its center of gravity by a method which he hopes will be found, less objectionable than that in which the composition and resolution of angular velocities are employed, and less complex than that given by Laplace and Poisson; he has also endeavoured to simplify the application of these equations to the case of the Earth.

In the First Part of these Tracts he has confined himself to the most prominent and important parts of each subject. In the Second Part, which will shortly be published, he intends, among other things, to give some account of the controversies which Laplace's Coefficients have given rise to; to investigate more fully the nature and properties of these functions; to give instances of their use in various problems; for this purpose to explain the mathematical theory of Electricity; to consider more particularly the Equations of motion of a rigid body about its center of gravity, and the conclusions that may be drawn from them; to give the theory of Jupiter's Satellites, and of Librations of the Moon; and to say something on the subject of Tides.

The Author has not given the investigation of the effect of the Earth's Oblateness on the motions of the Moon, but he has endeavoured to prove that this effect does not afford any additional evidence of the Earth's original Fluidity beyond that which may be obtained from the Figure of the Earth, and Law of Gravity.

MATHEMATICAL TRACTS,

PART I.

CHAPTER I.

FIGURE OF THE EARTH.

1. It has been well ascertained, by extensive and accurate geodetical measurements, that the general figure of the Earth is that of an oblate surface of revolution, described about the axis of diurnal rotation: and this fact suggests the idea, that the diurnal rotation may be in some way or other the cause of this peculiar figure, especially if we consider that the Sun and planets, which all rotate like the Earth, appear also to have the same sort of oblate form of revolution about their axes of rotation.

The most obvious and natural way of accounting for the influence thus apparently exerted on the figures of the planetary bodies by their rotation, is to suppose that they may once have been in a state of fluidity; for, conceive a fluid gravitating mass to be gradually put into a state of rotation round a fixed axis: it is evident that before the motion commenced it would, according to a well-known hydrostatical law, be arranged all through in concentric spherical strata of equal density; but on the motion of rotation commencing a centrifugal force would arise, which would be greater at greater distances from the axis, and would therefore evidently produce an oblateness in the forms of the strata, leaving them still symmetrical with respect to the axis. Thus the hypothesis of the original fluidity of the bodies of our system, considered in connection with their rotation, accounts for their oblate form.

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2. To account for the present solidity of the surface of our own planet, we may suppose that its temperature was originally so great as to keep it in a state of fusion, and that this was the cause of its fluidity; but that, in the course of ages, it, at least its surface, has cooled down and hardened into its present consistence. This supposition is borne out by geological facts; and it is by no means unlikely, if we consider that the principal body of our system is at present most probably in a state of fusion.

3. This hypothesis of the Earth's original fluidity receives much confirmation from observations on the intensity and direction of the force of gravity; for it follows from the hydrostatical law already alluded to, that the Earth, if fluid, ought to consist entirely of equidense strata of the same sort of form as the exterior surface*, and therefore the whole mass ought to be arranged symmetrically with respect to the axis of rotation, and nearly so with respect to the centre of that axis. Hence, the force of gravity, which is the resultant of the Earth's attraction and the centrifugal force, ought to be the same at all places in the same latitude, and nearly the same at all places in the same meridian.

Moreover it follows from another hydrostatical law, that the direction of this force of gravity ought to be every where perpendicular to the surface.

Now all this has been proved to be the case by numerous observations with pendulums, plumb-lines, levels, &c. (omitting very small variations, which may be easily accounted for in most cases). Hence, the hypothesis of the Earth's original fluidity is confirmed by the observations which have been made on the force of gravity.

4. But this hypothesis has been advanced almost to a moral certainty, by investigating precisely what effect it ought to have, if true, on the arrangement of the Earth's

* We suppose the Earth to be heterogeneous, because the pressure of the superincumbent mass must condense the central parts more than the superficial; besides, the well-known fact of the mean density of the whole Earth being greater than the density of the superficial parts, proves that the Earth is not homogeneous.

mass, and by comparing the result with observation; for it is found that if the hypothesis be true, the strata which compose the Earth ought to have not only an oblate form, but one very peculiar kind of oblate form; and it is found that this result admits of most satisfactory comparison with accurate and varied observation, and actually coincides with it in a most remarkable manner; from which we may conclude, almost with certainty, that the hypothesis is correct; for it is extremely difficult to account in any other way for so marked an agreement with observation of such a very peculiar result.

5. The object of the following pages is to give an account of this interesting investigation, and to state briefly the manner in which its result may be tested by observation. In the first place, we shall determine the law of arrangement of the Earth's mass, on the hypothesis of its original fluidity, by means of Laplace's powerful and beautiful Analysis; and in the next place, we shall deduce such results as shall admit of immediate comparison with observation. The most important of these results are; *The expression for the length of a meridian arc corresponding to a given difference of latitude, and, The law of variation of the force of gravity at different points of the Earth's surface.* The other results which we shall deduce depend on certain assumptions respecting the law of density of the Earth, and are therefore not so important. We now proceed, in the first place, to determine the law of arrangement of the Earth's mass, as follows.

6. *A heterogeneous fluid mass composed of particles which attract each other inversely as the square of the distance rotates uniformly in relative equilibrium* round a fixed axis: to determine the law of its arrangement.*

Take the axis of rotation as that of z, and let xyz be the co-ordinates of any particle δm, XYZ the resolved

* By *relative equilibrium* we mean that the particles of the mass, though actually moving, are at rest *relatively* to each other.

attractions of the mass on δm, ρ and p the density and pressure at the point (xyz), and ω the angular velocity of the mass. Then, by the principles of Hydrostatics, we have

$$dp = \rho \left\{ X dx + Y dy + Z dz + \omega^2 (x dx + y dy) \right\} \;\ldots\; (1)$$

To calculate the expression $(X dx + Y dy + Z dz)$, let $\delta m'$ be any attracting particle, and $x'y'z'$ its co-ordinates; then we have

$$X = \Sigma \frac{\delta m' (x' - x)}{\left\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\right\}^{\frac{3}{2}}}$$

and similar expressions for Y and Z.

Now assume (V) to denote the expression

$$\Sigma \frac{\delta m'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}},$$

i. e. *the sum of each particle divided by its distance from* δm. Then it is evident that

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz},$$

$$\text{and } X dx + Y dy + Z dz = \frac{dV}{dx} dx + \frac{dV}{dy} dy + \frac{dV}{dz} dz,$$

and therefore the equation (1) becomes

$$dp = \rho d \left\{ V + \frac{\omega^2}{2} (x^2 + y^2) \right\}$$

The coefficient of ρ here is a complete differential; hence by the principles of Hydrostatics, the necessary and sufficient conditions of equilibrium are, *that the whole mass be arranged in strata of equal density, the general equation to any one of them being*

$$C = V + \frac{\omega^2}{2} (x^2 + y^2),$$

C *being a constant different for different strata, the exterior surface being one of these strata, since it is a free surface.*

7. Hence the equations from which the problem is to be solved are

$$C = V + \frac{\omega^2}{2}(x^2 + y^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

$$V = \Sigma \frac{\delta m'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}} \ldots (B).$$

8. These equations are unfortunately very much involved in each other, so much so as to be scarcely manageable; for V must be found by integration between limits which depend on the form of the exterior stratum, and therefore on the equation (A); and also the law of density, and therefore the form of the internal strata, and therefore the equation (A), must be known in order to calculate V. But V is itself involved in (A), hence (A) cannot be made use of in calculating V. It will therefore be necessary to devise some way of eliminating V, without knowing what function it is. To do this in the general case is beyond the present powers of analysis; but in the particular case we are concerned with, the fact of the strata being all nearly spherical, introduces considerable simplification, and by using the ingenious analysis due to Laplace, we shall be able to eliminate V with comparative ease, at least, approximately, but with quite sufficient accuracy.

9. In the first place, the strata being nearly spherical, we shall find it convenient to make use of polar instead of rectangular co-ordinates, and we shall accordingly transform our equations as follows:

Let r, θ, ϕ, r', θ', ϕ', be the co-ordinates of δm and $\delta m'$ respectively; r, θ, ϕ, signifying the same as in Hymers' Geometry of three dimensions, page (77). Then we have

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r \cos\theta,$$

and similar expressions for x', y', z'.

Hence the equation (A) becomes

$$C = V + \frac{\omega^2}{2} r^2 \sin^2 \theta,$$

and the equation (B) becomes, observing that

$$\delta m' = \rho' \, r'^2 \sin \theta' \, dr' \, d\theta' \, d\phi'$$

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{r_1} \frac{\rho' r'^2 \sin \theta' \, dr' \, d\theta' \, d\phi'}{\sqrt{r^2 - 2rr' \{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')\} + r'^2}},$$

0 and r_1 being the limits of r', 0 and π of θ', and 0 and 2π of ϕ', r_1 being the value of r' at the surface, and therefore in general a function of θ' and ϕ'. It will of course on this account be necessary to integrate first with respect to r', but it is no matter in what order we perform the integrations relative to θ' and ϕ', since their limits are constants.

If these integrations could be performed, V would come out a function of r, θ, and ϕ, and unknown constants depending on the form of the strata and the law of density.

10. We shall find it convenient to put μ and μ' for $\cos \theta$ and $\cos \theta'$ respectively, this will give

$$\sin \theta' d\theta' = - d\mu',$$

and the limits of μ will be -1 and 1; or we may put $d\mu'$ instead of $-d\mu'$, if at the same time we reverse the limits of μ. Hence our equations become

$$C = V + \frac{\omega^2}{2} r^2 (1 - \mu^2) \dots\dots\dots\dots\dots\dots\dots (A'),$$

$$V = \int_0^{2\pi} \int_{-1}^{1} \int_0^{r_1} \frac{\rho' r'^2 dr' \, d\mu' d\phi'}{\sqrt{r^2 - 2rr'p + r'^2}} \dots\dots (B').$$

When for brevity we have put

$$\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') = p,$$

i. e. $\mu\mu' + \sqrt{1 - \mu^2} . \sqrt{1 - \mu'^2} . \cos (\phi - \phi') = p.$

11. We shall now introduce into these equations the condition that the strata are nearly spherical. If the strata were actually spherical, the whole mass would be symmetrical with respect to the centre, and therefore V being the sum of each particle divided by its distance from δm would depend simply on the distance of δm from the centre, and therefore be the same at all points of the same stratum. We may hence conclude, that if the strata instead of being actually spherical be only nearly so, V also, though not actually the same, will yet be nearly the same at all points of the same stratum. Now the value of V for any stratum is given by the equation (A') i. e.

$$V = C - \frac{\omega^2}{2} r^2 (1 - \mu^2);$$

but V (as we have shewn) ought to be nearly constant at all points of this stratum, hence the variable part of it, viz. $\frac{\omega^2}{2} r^2 (1 - \mu^2)$ must be always small: therefore since $r^2 (1 - \mu^2)$ is not always small, ω^2 must be so. We shall accordingly take ω^2 as the standard small quantity in our approximations, neglecting its square and higher powers in the first approximation.

12. Now ω^2 being a small quantity, we may suppose the equation to any nearly spherical surface, and therefore to any of the strata, to be put in the form

$$r = a(1 + \omega^2 u),$$

where a is the radius of any sphere which nearly coincides with the stratum (that sphere, suppose, which includes the same volume as the stratum*), and $a\omega^2 u$ is the small quantity to be added to a to make it equal to r, and therefore u in general will be some function of μ and ϕ. Moreover, u will be a function of a also, otherwise the strata would be all similar surfaces, which of course we have no right to assume them to be; a may be considered as the variable

* We make this supposition, at present, for the sake of giving a definite idea of what a is; hereafter it will be found an advantageous supposition.

parameter of the system of surfaces which the strata constitute. We shall introduce the variable a into our equations instead of r, and get every thing in terms of a, μ and ϕ, instead of r, μ, and ϕ; the advantages of this change will soon be perceived.

13. First, then, in the equation (A'), putting $a(1+\omega^2 u)$ instead of r, and neglecting the squares and higher powers of ω^2, we find

$$C = V + \frac{\omega^2}{2} a^2 (1 - \mu^2) \ldots (A'').$$

Next we shall make a similar substitution in the equation (B') by putting $r' = a'(1 + \omega^2 u')$, when u' denotes what u becomes when a, μ, and ϕ are exchanged for a', μ', and ϕ' respectively.

Assume for brevity,

$$\frac{r'^2}{\sqrt{r^2 - 2rr'p + r'^2}} = f(r, r'),$$

then in the equation (B') we shall have

$$\int_0^{r_1} \rho' f(r, r')\, dr' = \int_0^{a_1} \rho' f(r, r') \frac{dr'}{da'} \cdot da',$$

where a_1 is the parameter of the exterior surface,

and substituting for r', $f(r, r') = f\{r, (a' + a'\omega^2 u')\}$

$$= f(r, a') + \frac{d f(r, a')}{da'} a' \omega^2 u' + \&c.$$

and hence, since $\dfrac{dr'}{da'} = 1 + \omega^2 \cdot \dfrac{d(a'u')}{da'}$, we have

$$f(r,r')\frac{dr'}{da'} = f(r,a') + \omega^2 \left\{ f(r,a') \frac{d(a'u')}{da'} + \frac{d f(r,a')}{da'} a'u' \right\} + \&c.$$

$$= f(r, a') + \omega^2 \frac{d}{da'} \{ f(r, a') a' u' \} + \&c.$$

Hence, neglecting the squares and higher powers of ω^2, the equation (B') becomes

$$V = \int_0^{2\pi} \int_{-1}^{1} \int_0^{a_1} \rho' \left\{ f(r, a') + \omega^2 \frac{d}{da'} \{ f(r, a') a' u' \} \right\} da' d\mu' d\phi'.$$

14. This expression for V is much more manageable than that in the equation (B'); for in the first place the limits are all constant, and therefore we may take them in any order we please; and in the second place, ρ', instead of depending on all the variables, as of course it did before, is now a function of one variable only, viz. a'; for each stratum being equidense throughout, it is evident that ρ' is the same at every point of the same stratum, and therefore varies only when we pass from stratum to stratum, i. e. it varies with a' alone.

Hence, changing the order of the integrations, and bringing ρ' outside the integral signs relative to μ' and ϕ', we have

$$V = \int_0^{a_1} \rho' \left\{ \int_0^{2\pi} \int_{-1}^{1} f(r, a') \, d\mu' d\phi' \right.$$
$$\left. + \omega^2 \frac{d}{da'} \left(a' \int_0^{2\pi} \int_{-1}^{1} f(r, a') u' \, d\mu' \, d\phi' \right) \right\} da'.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots \quad (B'').$$

15. We have thus introduced into our equations the condition of the strata being nearly spherical.

We shall find it convenient to make a farther substitution in this equation, viz. by putting $r = a(1 + \omega^2 u)$, which, neglecting the squares and higher powers of ω^2, gives

$$V = \int_0^{a_1} \rho' \left\{ \int_0^{2\pi} \int_{-1}^{1} f(a, a') \, d\mu' \, d\phi' \right.$$
$$+ \omega^2 a u \frac{d}{da} \int_0^{2\pi} \int_{-1}^{1} f(a, a') \, d\mu' \, d\phi'$$
$$\left. + \omega^2 \frac{d}{da'} \left(a' \int_0^{2\pi} \int_{-1}^{1} f(a, a') u' \, d\mu' \, d\phi' \right) \right\} da'.$$
$$\dots\dots\dots\dots\dots\dots \quad (B''').$$

16. The next thing we shall do is to perform the integrations relative to μ' and ϕ'; to do this we shall expand the quantity $f(aa')$, which, since it represents

$$\frac{a'^2}{\sqrt{a'^2 - 2aa'p + a^2}},$$

may be expanded in a series of powers either of $\frac{a}{a'}$ or $\frac{a'}{a}$. The coefficients will evidently be the same whether we expand it in powers of $\frac{a}{a'}$ or of $\frac{a'}{a}$, they will in fact be the coefficients of the powers of h in the expansion of the quantity

$$\frac{1}{\sqrt{1 - 2ph + h^2}}.$$

We shall assume Q_0, Q_1, Q_2, &c. to denote these coefficients, i. e. we shall assume

$$\frac{1}{\sqrt{1 - 2ph + h^2}} = Q_0 + Q_1 h + Q_2 h^2 + \text{\&c.}$$

Q_0 will evidently be unity. The rest of these coefficients will be rational and integral functions of p, i.e. of

$$\mu\mu' + \sqrt{1 - \mu^2} \,.\, \sqrt{1 - \mu'^2} \,.\, \cos(\phi - \phi').$$

It is evident that they all become unity when p becomes unity; for then

$$\frac{1}{\sqrt{1 - 2ph + h^2}} \text{ becomes } \frac{1}{1 - h}, \text{ or } 1 + h + h^2 + \text{\&c.}$$

We shall have no occasion however to determine their forms. They (and other functions of the same character) are the celebrated coefficients of Laplace; they possess very remarkable properties, which we shall now digress to investigate, as they wonderfully facilitate the integrations we have to perform, and enable us to eliminate V from the equations (A'') and (B'''), with great facility, without knowing its form.

CHAPTER II.

LAPLACE'S COEFFICIENTS.

17. In order to investigate the properties of the functions Q_0, Q_1, Q_2, &c. introduced in the last chapter, we shall recur to the expression from which they were originally derived, viz.

$$\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}.$$

We meet with this expression constantly in physical problems, especially those in which attractions are concerned, and it is therefore worthy of particular consideration.

Assuming R to denote this expression, we have

$$R=\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$

and differentiating this equation twice relatively to xyz respectively,

$$\begin{aligned}\frac{dR}{dx}&=\frac{(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}\\&=R^3(x'-x)\\\frac{d^2R}{dx^2}&=3R^2\frac{dR}{dx}(x'-x)-R^3\\&=3R^5(x'-x)^2-R^3\end{aligned}$$

and similarly

$$\frac{d^2R}{dy^2}=3R^5(y'-y)^2-R^3$$

$$\frac{d^2R}{dz^2}=3R^5(z'-z)^2-R^3$$

hence evidently

$$\frac{d^2R}{dx^2}+\frac{d^2R}{dy^2}+\frac{d^2R}{dz^2}=3R^5\,\frac{1}{R^2}-3R^3$$
$$=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

18. We shall express this differential equation in terms of the polar co-ordinates $r\theta\phi$ instead of xyx. To facilitate the transformation we shall assume an auxiliary quantity s, such that

$$s = r\sin\theta;$$

and therefore since $x = r\sin\theta\cos\phi$, and $y = r\sin\theta\sin\phi$, we shall have

$$x = s\cos\phi,$$
$$y = s\sin\phi.$$

Then considering s and ϕ as independent variables instead of x and y, we have

$$\frac{dR}{ds}=\frac{dR}{dx}\,\frac{dx}{ds}+\frac{dR}{dy}\,\frac{dy}{ds}$$
$$=\frac{dR}{dx}\cos\phi+\frac{dR}{dy}\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

and $$\frac{d^2R}{ds^2}=\frac{d^2R}{dx^2}\cos^2\phi+2\,\frac{d^2R}{dx\,dy}\cos\phi\sin\phi+\frac{d^2R}{dy^2}\sin^2\phi \ldots (3).$$

and $$\frac{dR}{d\phi}=-\frac{dR}{dx}s\sin\phi+\frac{dR}{dy}s\cos\phi \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

$$\frac{d^2R}{d\phi^2}=\frac{d^2R}{dx^2}s^2\sin^2\phi-2\,\frac{d^2R}{dx\,dy}\,.\,s^2\cos\phi\sin\phi+\frac{d^2R}{dy^2}s^2\cos^2\phi$$
$$-\frac{dR}{dx}s\cos\phi-\frac{dR}{dy}s\sin\phi \ldots\ldots\ldots (5).$$

Equations (3) and (5) give

$$\frac{d^2R}{ds^2}+\frac{1}{s^2}\,\frac{d^2R}{d\phi^2}=\frac{d^2R}{dx^2}+\frac{d^2R}{dy^2}-\frac{1}{s}\left(\frac{dR}{dx}\cos\phi+\frac{dR}{dy}\sin\phi\right)$$
$$=\frac{d^2R}{dx^2}+\frac{d^2R}{dy^2}-\frac{1}{s}\,\frac{dR}{ds} \ldots\ldots\ldots\ldots\ldots (6),$$

by equation (2).

Now we have

$$z = r\cos\theta$$
$$s = r\sin\theta^*,$$

and these equations connect $z\,s\,r\,\theta$ in exactly the same way in which $x y s \phi$ are connected by the equations

$$x = s\cos\phi$$
$$y = s\sin\phi.$$

Hence we may prove exactly as before, that

$$\frac{dR}{dr^2} + \frac{1}{r^2}\frac{d^2R}{d\theta^2} = \frac{d^2R}{dz^2} + \frac{d^2R}{ds^2} - \frac{1}{r}\frac{dR}{dr};$$

adding this to the equation (6), we have by (1),

$$\frac{1}{s^2}\frac{d^2R}{d\phi^2} + \frac{d^2R}{dr^2} + \frac{1}{r^2}\frac{d^2R}{d\theta^2} = -\frac{1}{s}\frac{dR}{ds} - \frac{1}{r}\frac{dR}{dr}\ldots\ldots(7).$$

Now by (2) and (4),

$$\frac{dR}{ds}\sin\phi + \frac{dR}{d\phi}\frac{\cos\phi}{s} = \frac{dR}{dy},$$

and hence observing as before, that $z\,s\,r\,\theta$ are connected together in exactly the same way as $x\,y\,s\,\phi$, we have

$$\frac{dR}{dr}\sin\theta + \frac{dR}{d\theta}\frac{\cos\theta}{r} = \frac{dR}{ds};$$

hence, substituting this value of $\frac{dR}{ds}$ in equation (7), and putting $r\sin\theta$ for s, and multiplying by r^2, we have

$$\frac{d^2R}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dR}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2R}{d\phi^2} + r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = 0,$$

$$\text{or } \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dR}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2R}{d\phi^2} + \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = 0,$$

which is the equation (1) expressed in polar co-ordinates.

* The author finds that he has been anticipated in making this use of the auxiliary quantity s, by the Cambridge Mathematical Journal.

19. Now R expressed in polar co-ordinates becomes

$$\frac{1}{\sqrt{r'^2 - 2rr'p + r^2}},$$

$$\text{which} = \frac{1}{r'} \frac{1}{\sqrt{1 - 2p\dfrac{r}{r'} + \dfrac{r^2}{r'^2}}}$$

$$= \frac{1}{r'}\left\{Q_0 + Q_1\frac{r}{r'} + Q_2\frac{r^2}{r'^2} + \ldots + Q_n\frac{r^n}{r'^n} + \&c.\right\},$$

(See Art. 16). Hence substituting this value of R in the equation just obtained, and putting the coefficient of r^n equal to zero, since r is indeterminate, we find immediately

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dQ_n}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Q_n}{d\phi^2} + n(n+1)Q_n = 0,$$

$$\text{or, } \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_n}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2Q_n}{d\phi^2} + n(n+1)Q_n = 0,$$

which is a partial differential equation of the second order, connecting Q_n with μ and ϕ; of course, being such, it admits of an infinite number of solutions besides Q_n. We shall have no occasion to solve it, but we shall find it of use in investigating the properties of Q_n. All solutions of it which are rational and integral functions of $\cos\theta$, $\sin\theta$, $\cos\phi$, $\sin\phi$ (i. e. of μ, $\sqrt{1-\mu^2}$, $\cos\phi$, $\sin\phi$), are called Laplace's coefficients of the n^{th} order, having been first brought into notice by Laplace in his *Mécanique Céleste*, Liv. III.: the equation itself may be called Laplace's equation. Why we restrict Laplace's coefficients to be rational and integral functions of μ, $\sqrt{1-\mu^2}$, $\cos\phi$ and $\sin\phi$, will appear presently.

20. We may remark here that in consequence of the linearity of Laplace's equation, the sum of any number of

Laplace's coefficients of the n^{th} order is also a Laplace's coefficient of the n^{th} order.

Also any constant quantity is a Laplace's coefficient of the order 0, for if Y_0 be a Laplace's coefficient of the order 0, we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_0}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2Y_0}{d\phi^2}=0,$$

which equation is evidently satisfied by

$$Y_0 = \text{any constant},$$

and hence any constant is a Laplace's coefficient of the order 0.

It may easily be seen by trial that

$$a\mu, \text{ and } a\sqrt{1-\mu^2}\cos(\phi+\beta)$$

are Laplace's coefficients of the order 1, a and β being constants; and

$$a\left(\tfrac{1}{3}-\mu^2\right),\ a\mu\sqrt{1-\mu^2}\cos(\phi+\beta),\ a(1-\mu^2)\cos(2\phi+\beta),$$

are Laplace's coefficient of the order 2, and so on. We shall not have any occasion at present to determine the general expression for a Laplace's coefficient of the n^{th} order, but, to give clear ideas, we shall just state that it may be put in the form

$$A_0M_n+A_1(1-\mu^2)^{\frac{1}{2}}M_{n-1}\cos(\phi+a_1)+A_2(1-\mu^2)^{\frac{2}{2}}M_{n-2}\cos(2\phi+a_2),$$
$$\&\text{c.}\ldots\ldots\ldots\ldots +A_n(1-\mu^2)^{\frac{n}{2}}M_0\cos(n\phi+a_0).$$

When A_0 A_1 &c.... a_1, a_2 ... &c. are any constants, M_n M_{n-1}, &c. contain rational and integral functions of μ, of the dimensions n, $n-1$, $n-2$, &c. respectively *.

21. The first property we shall prove of Laplace's coefficients is this: If Y_m and Z_n be any two Laplace's coefficients of the m^{th} and n^{th} orders respectively, then

$$\int_0^{2\pi}\int_{-1}^{1} Y_m Z_n\, d\mu\, d\phi = 0, \text{ except when } m=n.$$

* We shall recur to this subject in Part II. of these Tracts.

For since Z_n satisfies Laplace's equation, we have

$$n(n+1)\int_0^{2\pi}\int_{-1}^{1} Y_m Z_n \, d\mu \, d\phi$$

$$= -\int_0^{2\pi}\int_{-1}^{1} Y_m \left\{\frac{d}{d\mu}\left((1-\mu^2)\frac{dZ_n}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2 Z_n}{d\phi^2}\right\} d\mu \, d\phi \quad (1).$$

Now integrating by parts, and observing that $1-\mu^2 = 0$, at each limit we have

$$\int_{-1}^{1} Y_m \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dZ_n}{d\mu}\right\} d\mu = -\int_{-1}^{1} \frac{dY_m}{d\mu}(1-\mu^2)\frac{dZ_n}{d\mu}\, d\mu,$$

and similarly,

$$\int_0^{2\pi} Y_m \frac{d^2 Z_n}{d\phi^2}\, d\phi = -\int_0^{2\pi} \frac{dY_m}{d\phi}\frac{dZ_n}{d\phi}\, d\phi,$$

*observing that $Y_m \dfrac{dZ_n}{d\phi}$ is the same at each limit, because Y_m and Z_n are functions of $\sin\phi$ and $\cos\phi$, and not of ϕ simply; hence substituting in (1)

$$n\,.\,(n+1)\int_0^{2\pi}\int_{-1}^{1} Y_m Z_n \, d\mu \, d\phi$$

$$= \int_0^{2\pi}\int_{-1}^{1} \left\{(1-\mu^2)\frac{dY_m}{d\mu}\cdot\frac{dZ_n}{d\mu} + \frac{1}{1-\mu^2}\frac{dY_m}{d\phi}\cdot\frac{dZ_n}{d\phi}\right\} d\mu \, d\phi.$$

In exactly the same way we may shew that

$$m\,.\,(m+1)\int_0^{2\pi}\int_{-1}^{1} Y_m Z_n \, d\mu \, d\phi$$

$$= \int_0^{2\pi}\int_{-1}^{1} \left\{(1-\mu^2)\frac{dY_m}{d\mu}\cdot\frac{dZ_n}{d\mu} + \frac{1}{1-\mu^2}\frac{dY_m}{d\phi}\frac{dZ_n}{d\phi}\right\} d\mu \, d\phi\,;$$

* This is the reason why we have assumed Laplace's coefficients to be functions of $\sin\phi$ and $\cos\phi$, and not of ϕ simply.

Hence subtracting

$$\{n(n+1) - m(m+1)\} \int_0^{2\pi} \int_{-1}^{1} Y_m Z_n \, d\mu \, d\phi = 0.$$

Now the factor $n(n+1) - m(m+1)$ does not $= 0$, except when $m = n$; hence the other factor must be zero, hence

$$\int_0^{2\pi} \int_{-1}^{1} Y_m Z_n \, d\mu \, d\phi = 0,$$

except when $m = n$.

22. Since $Q_0 = 1$ (see Art. 16), we have

$$\int_0^{2\pi} \int_{-1}^{1} Q_n \, d\mu \, d\phi = \int_0^{2\pi} \int_{-1}^{1} Q_n Q_0 \, d\mu \, d\phi$$

$$= 0, \text{ when } n \text{ is greater than } 0,$$

$$= \int_0^{2\pi} \int_{-1}^{1} d\mu \, d\phi, \text{ when } n = 0$$

$$= 4\pi.$$

It need scarcely be remarked that Q_0, Q_1, Q_2, &c. possess exactly the same properties with respect to μ' and ϕ' that they do with respect to μ and ϕ.

23. We shall now have occasion to introduce a remarkable discontinuous function, but before we do so we shall give a simple example of functions of this description, in order to render our reasoning more satisfactory to those who have not been accustomed to them.

We may easily prove, by the aid of the exponential value of the cosine of an arc, that

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) \text{ ad inf.} \ldots = -\frac{\sin\left(\alpha - \frac{\beta}{2}\right)}{2 \sin \frac{\beta}{2}};$$

suppose here that $\alpha = \frac{\beta}{2}$, then we find

$$\cos \alpha + \cos 3\alpha + \cos 5\alpha + \ldots = -\frac{0}{2 \sin \alpha};$$

hence this series is always zero, except when $a =$ any multiple of π, in which case $\sin a$ becomes zero, and therefore the series becomes $\frac{0}{0}$; thus though each term of this series varies continuously with a, the series itself varies discontinuously, being constantly zero, except when a passes through any of the values $0, \pm\pi, \pm 2\pi$, &c., when the series suddenly becomes $\frac{0}{0}$; i. e., some unknown or indeterminate quantity. To explain the nature of this series more clearly, we observe that

$$\sin\frac{\beta}{2}\left\{\cos a + \cos(a+\beta) + \&c.\right\} = -\tfrac{1}{2}\sin\left(a - \frac{\beta}{2}\right),$$

whatever be the values of a and β; suppose $a = \frac{\beta}{2}$; then $\sin\left(a - \frac{\beta}{2}\right)$ becomes zero, whatever be the value of a; hence

$$\sin a\left\{\cos a + \cos 3a + \cos 5a + \&c.\right\} = 0,$$

for all values of a.

Now as long as a is not a multiple of π, $\sin a$ will not be zero, and therefore

$$\cos a + \cos 3a + \ldots \&c.$$

must be zero; but if a be any multiple of π, then $\sin a$ will be zero, and our equation will be satisfied quite independently of its other factor, and hence will give us no information as to the value of that factor; hence when a is any multiple of π, $\cos a + \cos 3a + \&c.$ is some unknown or indeterminate quantity.

It is important to remark that the change in the value of $\cos a + \cos 3a + \&c.$, when a becomes a multiple of π, is perfectly sudden, for since the second member of the equation is always absolutely zero, it is evident that as long as $\sin a$ is not actually zero, though it differs from it by ever so

small a quantity, $\cos a + \cos 3a +$ &c.... must be so; for this reason $\cos a + \cos 3a +$ &c. is called a discontinuous function.

24. We shall now bring forward the remarkable discontinuous function we alluded to: it is the following series, viz.

$$Q_0 + 3Q_, + \text{\&c.} + (2n+1)\,Q_n + \text{\&c. ad inf.}$$

this series is of exactly the same nature as that we have just considered, being a function of the variables μ and ϕ, which is always zero, except for certain particular values of these variables.

To shew this, we have

$$Q + Q_,h + \dots Q_n h^n + \text{\&c.} = \frac{1}{\sqrt{1 - 2ph + h^2}},$$

and differentiating this relatively to h and multiplying by $2h$,

$$2Q_,h + \dots 2nQ_nh^n + \dots \text{\&c.} = \frac{2ph - 2h^2}{(1 - 2ph + h^2)^{\frac{3}{2}}},$$

and adding these equations,

$$Q_0 + 3Q_,h \dots (2n+1)\,Q_nh^n + \text{\&c.} = \frac{1 - h^2}{(1 - 2ph + h^2)^{\frac{3}{2}}};$$

now here put $h = 1$ and we find

$$Q_0 + 3Q_, + \text{\&c.} = \frac{0}{(2 - 2p)^{\frac{3}{2}}},$$

hence $Q_0 + 3Q_, +$ &c. $= 0$ for all values of p, except $p = 1$, when it becomes $\frac{0}{0}$; now

$$p = \mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\phi - \phi'),$$

hence if $p = 1$,

$$\cos(\phi - \phi') = \frac{1 - \mu\mu'}{\sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}},$$

but $\cos(\phi - \phi')$ is not greater than 1;

hence,

$$1 - \mu\mu' \text{ is not } > \sqrt{1-\mu^2}\sqrt{1-\mu'^2};$$

or, squaring and reducing,

$$(\mu - \mu')^2 \text{ is not } > 0,$$

which cannot be unless $\mu = \mu'$, and this will give

$$\cos(\phi - \phi') = 1;$$

and therefore $\phi \sim \phi'$ is zero, or some multiple of 2π; hence the series $Q_0 + 3Q_1 +$ &c. is a discontinuous function, being always zero, except when $\mu = \mu'$ and $\phi \sim \phi' = 2m\pi$, (m being any integer,) in which case it becomes $\frac{0}{0}$. It is evident, as in the former case, that this series is perfectly discontinuous, being zero for all values of p that differ even in the *least* degree from unity, and then when $p = 1$, suddenly assuming the form $\frac{0}{0}$.

25. Now, wherever we have occasion to use the series $Q_0 + 3Q_1 +$ &c., it will occur under integral signs relative to μ and ϕ, and the limits of ϕ will be 0 and 2π; hence, by what we have proved in the note* respecting the limits of

* If x be any arbitrary quantity occurring in any investigation, its differential dx may be defined to be any small increment of x, made use of with the understanding that it is to be put equal to zero at the end of the investigation; and if $f(x)$ be any function of x, its differential $df(x)$ may be defined to be the corresponding increment of $f(x)$, that is,

$$df(x) = f(x + dx) - f(x).$$

The symbol $\int$ written before a differential, is generally taken to denote the quantity from which the differential is derived, that is,

$$\int df(x) = f(x),$$

and the notation $\int_{x_1}^{x_2} df(x)$ is taken to denote the difference $f(x_2) - f(x_1)$: but this notation has a much more important signification: for in the equation

$$df(x) = f(x + dx) - f(x),$$

put for x successively the values x_1, $x_1 + dn$, $x_1 + 2dn$, &c., $x_1 + (n-1)dx$, and add the results, and we find

$$df(x_1) + df(x_1 + dx) + df(x_1 + 2dx) \text{ \&c.} + df\{x_1 + (n-1)dx\}$$
$$= f(x_1 + ndx) - f(x_1),$$

or

integrals, ϕ will receive all the values between 0 and 2π, inclusive of the inferior limit, and *exclusive* of the superior, and will therefore never actually be equal to, or exceed 2π. Also in all cases we shall be concerned with the same may be supposed true of ϕ'; for in all our investigations, wherever ϕ' recurs, the value $\phi' = 2\pi$, or any greater quantity, will be only a repetition of $\phi' = 0$, or some value between 0 and 2π; hence we may consider that $\phi \sim \phi'$ never actually equals or exceeds 2π; and hence it will be only for one value of $\phi \sim \phi'$, viz. 0, that p will become unity; hence, by what we have proved, the series

$$Q_0 + 3\,Q_1 + \&c.$$

in all cases we shall be concerned with, will be absolutely zero for all values of μ and ϕ, except the single values $\mu = \mu'$ and $\phi = \phi'$. We now proceed to prove some remarkable properties of this series, which result from its discontinuous nature.

26. From Art. 22 it appears immediately that

$$\int_0^{2\pi}\int_{-1}^{1}\{Q_0 + 3\,Q_1 + \&c.\}\,d\mu\,d\phi = \int_0^{2\pi}\int_{-1}^{1} Q_0\,d\mu\,d\phi$$
$$= 4\pi.$$

Now, here the quantity under the integral signs is, as we have proved, always zero, except when $\mu = \mu'$ and $\phi = \phi'$; it is therefore no matter what the limits of the integration be, provided they include between them the

or supposing $x_1 + n\,dx = x_2$,

$$df(x_1) + df(x_1 + dx) + \&c. \text{ till we come to } df(x_2 - dx) = f(x_2) - f(x_1)$$
$$= \int_{x_1}^{x_2} df(x).$$

Hence it appears that $\int_{x_1}^{x_2} df(x)$ denotes the sum of a series of values of $df(x)$, got by giving x all its values between the limits x_1 and x_2 *inclusive* of the former limit and *exclusive* of the latter; that is to say, all the values of x which form an arithmetic series whose common difference is dx, commencing with x_1 and ending with $x_2 - dx$. The remark respecting the limits is important whenever discontinuous functions are concerned, as in our present investigation; and we must remember that though the last value of x approaches indefinitely near to x_2, it never actually becomes equal to it.

values ($\mu = \mu'$) and ($\phi = \phi'$), respectively; hence, if μ_1, μ_2, ϕ_1, ϕ_2*, be any limits which do this, we have

$$\int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} (Q_0 + 3\,Q_1 + \&c. \ldots)\, d\mu\, d\phi = 4\,\pi.$$

27. In the same manner, if $F(\mu\phi)$ be any function of μ and ϕ, which is always finite between the limits -1 and 1, 0 and 2π, $F(\mu\phi)(Q_0 + 3\,Q_1 + \&c.)$ will be always zero, except when $\mu = \mu'$ and $\phi = \phi'$, and we shall have, as before,

$$\int_0^{2\pi} \int_{-1}^{1} F(\mu\,\phi)\,\{Q_0 + 3\,Q_1 + \&c.\}\, d\mu\, d\phi$$

$$= \int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} F(\mu\phi)\,\{Q_0 + 3\,Q_1 + \ldots\}\, d\mu\, d\phi.$$

28. Now let $F(\mu''\phi'')$ be the greatest value of $F(\mu\phi)$, between the limits μ_1, μ_2, ϕ_1, ϕ_2; and let $F(\mu_{\prime\prime}\phi_{\prime\prime})$ be the least; then it is evident from the nature of an integral, considered as a sum, that

$$\int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} F(\mu'\phi')\,(Q_0 + 3\,Q_1 + \&c)\, d\mu\, d\phi$$

is not greater than

$$F(\mu''\phi'') \int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} (Q_0 + 3Q_1 + \ldots)\, d\mu\, d\phi,$$

and not less than

$$F(\mu_{\prime\prime}\phi_{\prime\prime}) \int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} (Q_0 + 3\,Q_1 - \ldots)\, d\mu\, d\phi,$$

i. e. (by Art. 22),

not greater than $4\,\pi\, F(\mu''\phi'')$,

and not less than $4\,\pi\, F(\mu_{\prime\prime}\phi_{\prime\prime})$;

and this is true, no matter how close together the limits

* Of course these limits are supposed to be included between -1 and $+1$, 0 and 2π.

$\mu_1, \mu_2, \phi_1, \phi_2$, be taken, provided μ' and ϕ' be included between them. Now μ'', ϕ'', and $\mu_{''}, \phi_{''}$, are also always included between these limits; hence, since $\mu'\phi'$, μ'', ϕ'', $\mu_{''}, \phi_{''}$, are respectively always included between limits which we may take as close together as we please, it is evident that we may suppose μ'', ϕ'', and $\mu_{''}, \phi_{''}$, to differ from $\mu'\phi'$ respectively by as small quantities as we please; and therefore, since $F(\mu\phi)$ is always finite, we may in the above inequalities suppose $F(\mu'', \phi'')$ and $F(\mu_{''}, \phi_{''})$ as nearly equal to $F(\mu'\phi')$ as we please, which evidently cannot be, unless

$$\int_{\phi_1}^{\phi_2}\int_{\mu_1}^{\mu_2} F(\mu\phi)\,(Q_0 + 3\,Q_1 + \ldots)\,d\mu\,d\phi = 4\pi F(\mu'\phi').$$

29. We shall give another demonstration of this remarkable result.

Assume, as we evidently may,

$$\int_0^{2\pi}\int_{-1}^{1} F(\mu\phi)\,(Q_0 + 3\,Q_1 + \ldots)\,d\mu\,d\phi = f(\mu'\phi').$$

Then, as before,

$$\int_{\phi_1}^{\phi_2}\int_{\mu_1}^{\mu_2} F(\mu\phi)\,(Q_0 + 3\,Q\ \ldots)\,d\mu\,d\phi = f(\mu'\phi');$$

multiply this equation by $d\mu'\,d\phi'$, and integrate between the same limits, and we have*

$$\int_{\phi_1}^{\phi_2}\int_{\mu_1}^{\mu_2} F(\mu\phi)\left\{\int_{\phi_1}^{\phi_2}\int_{\mu_1}^{\mu_2}(Q_0 + 3\,Q_1 + \ldots)\,d\mu'd\phi'\right\}d\mu\,d\phi$$
$$= \int_{\phi_1}^{\phi_2}\int_{\mu_1}^{\mu_2} f(\mu'\phi')\,d\mu'\,d\phi'.$$

* It is necessary to take the same limits, otherwise in the integral

$$\int_{\phi_1}^{\phi_2}\int_{\mu_1}^{\mu_2}(Q_0 + 3\,Q_1 + \ldots)\,d\mu'\,d\phi',$$

μ and ϕ, the values of the variables for which $Q_0 + 3\,Q_1 + \ldots$ becomes $\frac{0}{0}$, will not be always included between the limits, and therefore Art. 26 will not apply to it, and our proof will be incorrect.

Hence, by Art. 26,

$$4\pi \int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} F(\mu\phi)\, d\mu\, d\phi = \int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} f(\mu'\phi')\, d\mu'\, d\phi',$$

differentiating this equation relatively to ϕ_2 and μ_2 successively*, we have

$$4\pi F(\mu_2\phi_2) = f(\mu_2\phi_2);$$

hence, since μ_2 and ϕ_2 are arbitrary, $f = 4\pi F$; and therefore

$$\int_{\phi_1}^{\phi_2} \int_{\mu_1}^{\mu_2} F(\mu\phi)\,(Q_0 + 3Q_1 + \ldots)\, d\mu\, d\phi = 4\pi F(\mu'\phi').$$

30. We may hence find the value of

$$\int_0^{2\pi} \int_{-1}^{1} Y_n Q_n\, d\mu\, d\phi.$$

Y_n being any Laplace's coefficient; for Y_n being a rational and integral function of μ, $\sqrt{1-\omega^2}$, $\cos\phi$, $\sin\phi$† will be always finite; hence we may put Y_n for $F(\mu\phi)$ in Art. 28, and we find immediately by Art. 21,

$$\int_0^{2\pi} \int_{-1}^{1} Y_n Q_n\, d\mu\, d\phi = \frac{4\pi}{2n+1} Y',$$

* To shew how to differentiate a definite integral with respect to its limits, let $f(x) + C$ denote the indefinite integral of $f'(x)$, then

$$\int_{x_1}^{x_2} f'(x)\, dx = f(x_2) - f(x_1),$$

differentiating this relative to x_2, we have

$$\frac{d}{dx_2} \int_{x_1}^{x_2} f'(x)\, dx = \frac{df(x_2)}{dx_2} = f'(x_2),$$

and differentiating relative to x_1,

$$\frac{d}{dx_1} \int_{x_1}^{x_2} f'(x)\, dx = -\frac{df(x_1)}{dx_1} = -f'(x_1);$$

and in a similar manner we may differentiate integrals relative to two or more variables.

† This is the reason why we have restricted Laplace's coefficients to be rational and integral.

where Y' denotes what Y becomes, when μ' and ϕ' are put for μ and ϕ in it.

This is a very important result; in fact, this, and that in Art. 21, are the properties which render Laplace's coefficients so very useful in integrations such as we have to perform in Art. 15.

31. The equation deduced in Art. 28, interchanging μ' and ϕ' for μ and ϕ, shews that if $F(\mu\phi)$ be any function of μ and ϕ, which is always finite, it may be expanded in a series of Laplace's coefficients; for by this equation $F(\mu\phi)$ = a series whose general term is

$$\frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} F(\mu'\phi')\, Q_n\, d\mu'\, d\phi'.$$

Now this quantity evidently satisfies any linear differential equation relative to μ and ϕ that Q_n satisfies; therefore it satisfies Laplace's equation of the n^{th} order; moreover it is a rational and integral function of μ, $\sqrt{1-\mu^2}$, $\cos\phi$, $\sin\phi$, for Q_n is so, and

$$\frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} F(\mu'\phi')\ Q_n\, d\mu'\, d\phi'$$

will evidently differ from Q_n, considered as a function of μ, $\sqrt{1-\mu^2}$, $\cos\phi$, $\sin\phi$, only in having different coefficients to the powers of these quantities; that is to say, if A' be any coefficient in Q_n, then the corresponding coefficient in

$$\frac{2n+1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} F(\mu'\phi')\ Q_n\, d\mu'\, d\phi'$$

will be

$$\frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} F(\mu'\phi')\,.\,A'\, d\mu'\, d\phi'\,;$$

hence the several terms of the series to which $F'(\mu'\phi')$ is equivalent are rational and integral functions of μ, $\sqrt{1-\mu^2}$, $\cos\phi$, and $\sin\phi$, which satisfy Laplace's equation, and

are therefore, according to our definition, Laplace's coefficients.

32. No function can be expanded in more than *one* series of Laplace's coefficients.

For, if possible, let

$$Y_0 + Y_1 + Y_2 + \dots + \text{\&c. and } Z_0 + Z_1 + Z_2 + \text{\&c.}$$

be two different series of Laplace's coefficients equivalent to the same function, then, since this is the case, we have

$$Y_0 - Z_0 + Y_1 - Z_1 + Y_2 - Z_2 + \text{\&c.} = 0;$$

multiplying this equation by $Q_n \, d\mu \, d\phi$, and integrating between the limits $-1, 1$; $0, 2\pi$, we find (by Art. 22 and 30),

$$\frac{4\pi}{2n+1}(Y'_n - Z'_n) = 0,$$

hence $Y_n = Z_n$; and therefore the series are the same, and the function can be expanded only in *one* series of Laplace's coefficients.

33. The conclusion we have just arrived at seems to be at variance with the fact that the quantity $\dfrac{1}{\sqrt{1-2ph+h^2}}$ may be expanded in two distinct series of Laplace's coefficients, viz.

$$Q_0 + Q_1 h + Q_2 h^2 + \dots$$

$$\text{and } Q_0 \frac{1}{h} + Q_1 \frac{1}{h^2} + Q_2 \frac{1}{h^3} + \dots$$

but this is only an apparent discrepancy, for $\dfrac{1}{\sqrt{1-2ph+h^2}}$, the function to be developed, admits of two values, one positive and the other negative, on account of the ambiguity of the sign of the square root; and therefore ought to admit of two developments. That the above are the two developments corresponding to the two values of the square root, will follow from the following proposition: viz.

34. To determine the sign of the series

$$Q_0 + Q_1 h + Q_2 h^2 + \dots$$

This series being the development of the expression

$$\frac{1}{\sqrt{1 - 2ph + h^2}},$$

it is evident that it can never become 0 as long as h is not infinite, but it will become infinite when

$$1 - 2ph + h^2 = 0;$$

$$\text{that is, when } p = \frac{1 + h^2}{2h}.$$

Now $1 + h^2$ is always greater than $2h$, except when $h = 1$, in which case $1 + h^2 = 2h$; hence, since p, being a cosine, can never exceed unity, this equation can only be satisfied by $h = 1$, and therefore $p = 1$; hence

$$Q_0 + Q_1 h + Q_2 h^2 + \dots$$

cannot become infinity unless h be unity; hence, if h be not infinity or unity, this series can never become zero nor infinity, and therefore can never change its sign: supposing then that h is not infinity or unity,

$$Q_0 + Q_1 h + Q_2 h^2 + \dots$$

will always have the same sign whatever be the value of p. Hence, supposing $p = 1$, and therefore $Q_0 = 1$, $Q_1 = 1$, &c. (Art. 16.) it is evident that

$$Q_0 + Q_1 h + \text{\&c. has the same sign as } 1 + h + h^2 \dots$$

$$\text{i. e. as } \frac{1}{1 - h};$$

hence, if h be less than unity, this series is positive, and if h be greater than unity, it is negative.

In the same way it may be proved that

$$\frac{1}{h}\left\{Q_0 + Q_1 \frac{1}{h} + Q_2 \frac{1}{h^2} + \right\} \text{ has the same sign as } \frac{1}{h - 1},$$

and is therefore always negative when h is less than unity, and positive when h is greater than unity.

Hence, when h is less than unity,

$$Q_0 + Q_1 h + Q_2 h^2 + \text{ is the positive value of } \frac{1}{\sqrt{1 - 2ph + h^2}},$$

$$\text{and } \frac{1}{h}\left\{Q_0 + Q_1 \frac{1}{h} + \dots\right\} \text{ the negative value;}$$

and when h is greater than unity, the reverse is the case.

35. To represent, then, the true general development of $\dfrac{1}{\sqrt{1 - 2ph + h^2}}$, let k be a discontinuous function of h,

such that $k = 1$, when h is less than unity;

and $k = 0$, when h is greater than unity.

Then, whatever h be, the positive value of $\dfrac{1}{\sqrt{1-2ph+h^2}}$, will evidently be

$$k\left\{Q_0 + Q_1 h + Q_2 h^2 + \dots\right\} + (1 - k)\left\{Q_0 \frac{1}{h} + Q_1 \frac{1}{h^2} + \dots\right\};$$

and the negative value will be

$$(1 - k)\left\{Q_0 + Q_1 h + Q_2 h^2 + \dots\right\} + k\left\{Q_0 \frac{1}{h} + Q\,\frac{1}{h^2} + \dots\right\}.$$

Thus, in reality, $\dfrac{1}{\sqrt{1 - 2ph + h^2}}$, if we restrict ourselves to only one of its values, can be developed in only one series of Laplace's coefficients. The conclusion we have just arrived at respecting the true development of this quantity, will be highly important in our future investigations.

36. We shall conclude this chapter with the following important property of Laplace's coefficients, viz.

If $$Y_0 + Y_1 + Y_2 + \&c. = 0 \ \ldots\ldots\ldots\ldots\ldots \ (1),$$

be any equation arranged in a series of Laplace's coefficients, μ and ϕ being indeterminate, then we must have

$$Y_0 = 0, \quad Y_1 = 0, \quad Y_2 = 0, \quad \&c.$$

for, multiplying (1) by $Q_n \, d\mu \, d\phi$, and integrating between the limits -1, 1, and 0, 2π, we have, (by Arts. 22 and 30,)

$$\frac{4\pi}{2n+1} Y_n' = 0\,;$$

and therefore $Y_n = 0$.

We have now considered these remarkable functions at sufficient length for our present purpose; we shall recur to this subject in Part II. of these Tracts.

CHAPTER III.

FIGURE OF THE EARTH.

37. We are now prepared to return to the equation (B'''), see Art. 15; the properties we have proved Laplace's coefficients to possess, will enable us to perform the integrations relative to μ' and ϕ' with great facility.

In Art. 16 we stated that we should expand the quantity $f(a, a')$, or $\dfrac{a'^2}{\sqrt{a^2 - 2aa'p + a'^2}}$, in either of the series

$$\frac{a'^2}{a}\left\{Q_0 + Q_1\frac{a'}{a} + \&c.\right\}, \text{ or } a'\left\{Q_0 + Q_1\frac{a}{a'} + \&c.\right\}.$$

Now there is an ambiguity in this quantity, since on account of the square root, it admits of two values, one positive and the other negative; but, on referring to Art. 6, it is evident that the square root is always supposed to have its positive value; for the distance between δm and $\delta m'$, which is expressed by this square root, is evidently taken to be the absolute or numerical distance, without reference to sign; hence, by what has been proved in Art. 35, neither of the above series will give the true general development of $f(a, a')$, but we must put

$$f(a, a') = k \cdot \frac{a'^2}{a}\left\{Q_0 + Q_1\frac{a'}{a} + Q_2\frac{a'^2}{a^2} + \&c.\right\}$$
$$+ (1 - k)a'\left\{Q_0 + Q_1\frac{a}{a'} + Q_2\frac{a^2}{a'^2} + \&c.\right\},$$

where k is a discontinuous function of $\dfrac{a'}{a}$, which is always

unity while a' is less than a, and zero when a' is greater than a. For brevity we shall take A_n to denote the coefficient of Q_n in this series; i.e.

$$A_n = k \cdot \frac{a'^{n+2}}{a^{n+1}} + (1-k)\frac{a^n}{a'^{n-1}};$$

and then, we shall have

$$f(a,a') = A_0 Q_0 + A_1 Q_1 + A_2 Q_2 + \&c.$$

38. What we have proved of Laplace's coefficients naturally suggests the advantage of expanding the quantity u' in a series of Laplace's coefficients; this may be done, since, on account of the assumption we have made respecting the nearly spherical form of the strata, u' can never become infinite; we shall assume, therefore,

$$u' = u'_0 + u'_1 + u'_2 + \&c.$$

u'_0, u'_1, &c. being Laplace's coefficients of the orders 0, 1, 2, &c. and functions of a', μ', and ϕ'.

We shall denote by u_0, u_1, u_2, &c. what u'_0, u'_1, u'_2, &c.... become when a, μ, and ϕ are substituted for a', μ', ϕ', and therefore we shall have

$$u = u_0 + u_1 + u_2 + \&c.$$

We shall also have occasion to make use of the values of u'_0, u'_1, u'_2, &c. when μ and ϕ alone are put for μ' and ϕ', a' remaining unaltered. These values we shall denote by $'u_0$, $'u_1$, $'u_2$, &c.

39. Hence in the equation (B'''), see Art. 15, we find immediately by Arts. 22 and 30,

$$\int_0^{2\pi}\int_{-1}^{1} f(a,a')\, d\mu'\, d\phi' = 4\pi A^0,$$

and

$$\int_0^{2\pi}\int_{-1}^{1} f(a,a')\, u'\, d\mu'\, d\phi' = 4\pi \left\{ A_0{}'u_0 + \tfrac{1}{3} A_1{}'u_1 \right.$$

$$\left. + \dots \frac{1}{2n+1} A_n{}'u_n + \dots \right\};$$

and hence, putting $u_0 + u_1 + u_2$, &c. for u,

$$V = 4\pi \int_0^{a_1} \rho' \left\{ A_0 + \omega^2 a \left(u_0 + u_1 + u_2 + \&c.\right) \frac{dA_0}{da} \right.$$

$$\left. + \omega^2 \frac{d}{da'} \left\{ a' \left(A_0' u_0 + \tfrac{1}{3} A_1' u_1 + \&c. \right) \right\} \right\} da'$$

$$= 4\pi \int_0^{a_1} \rho' A_0 da$$

+ a series whose general term is

$$4\pi \omega^2 \int_0^{a_1} \rho' \left\{ a u_n \frac{dA_0}{da} + \frac{1}{2n+1} \frac{d}{da'} (a' A_n' u_n) \right\} da'.$$

40. If we put for A_0 and A_n their values, viz.

$$k \frac{a'^2}{a} + (1-k) a', \text{ and } k \frac{a'^{n+2}}{a^{n+1}} + (1-k) \frac{a^n}{a'^{n-1}},$$

(see Art. 37.); the factor of $4\pi\omega^2$ in this general term becomes

$$\int_0^{a_1} \rho' \left\{ -\frac{u^n}{a} k a'^2 + \frac{1}{(2n+1) a^{n+1}} \frac{d}{da'} (k a'^{n+3}{}' u_n) \right.$$

$$\left. + \frac{a^n}{2n+1} \frac{d}{da'} \left((1-k) \frac{u'_n}{a'^{n-2}} \right) \right\} da',$$

Now k is always unity while a' varies between the limits 0 and a, and zero while a' varies between the limits a and a_1 (a being of course never greater than a_1); hence that part of the quantity under the integral sign which is multiplied by k will not exist except between the limits 0 and a, and that part multiplied by $1-k$ will not exist except between the limits a and a_1; hence this integral may evidently be put in the form

$$-\frac{u_n}{a} \int_0^a \rho' a'^2 da' + \frac{1}{(2n+1) a^{n+1}} \int_0^a \rho' \frac{d}{da'} ({}'u_n a'^{n+3}) da$$

$$+ \frac{a^n}{2n+1} \int_a^{a_1} \rho' \frac{d}{da'} \left(\frac{{}'u_n}{a'^{n-2}} \right) da'.$$

We shall, for brevity, denote this expression by

$$\sigma_n \ u_n,$$

σ_n being merely a prefix assumed to express a certain operation performed on u_n.

In the same way the first term of V, viz.

$$4\pi \int_0^{a_1} \rho' A_0 \, d a',$$

will evidently become

$$4\pi \int_0^{a_1} \rho' \left\{ k \frac{a'^2}{a} + (1-k) a' \right\} d a',$$

or

$$4\pi \int_0^{a} \rho' a'^2 \, d a' + 4\pi \int_a^{a_1} \rho' a' \, d a'.$$

Hence the development of V becomes finally

$$V = \frac{4\pi}{a} \int_0^{a} \rho' a'^2 \, d a' + 4\pi \int_a^{a_1} \rho' a' \, d a'$$
$$+ 4\pi \omega^2 \left\{ \sigma_0 u_0 + \sigma_1 u_1 + \sigma_2 u_2 + \ \&c. \right\}.$$

It is evident that this is a series of Laplace's coefficients, the sum of the first three terms being a coefficient of the order 0, and the succeeding terms of the order 1, 2, 3, &c. respectively.

41. We shall now express the value of V got from the equation (A'') (see Art. 13) in Laplace's coefficients; to do this we observe, by trial, that $\frac{1}{3} - \mu^2$ is a Laplace's coefficient of the order 2; hence the value of V got from (A'') will be arranged in Laplace's coefficients by simply putting it in the form

$$V = C - \frac{\omega^2}{2} a^2 \left(\tfrac{2}{3} + \tfrac{1}{3} - \mu^2 \right),$$

or

$$V = C - \frac{\omega^2 a^2}{3} - \frac{\omega^2 a^2}{2} \left(\tfrac{1}{3} - \mu^2 \right).$$

42. If we now subtract the two equations we have thus obtained, we shall eliminate V and arrive at an equation consisting of a series of Laplace's coefficients, which, by (Art. 36), must be put separately equal to zero; hence we get the following equations, viz.

$$\frac{4\pi}{a}\int_0^a \rho' a'^2 da' + 4\pi\int_a^{a_1} \rho' a' da' + 4\pi\omega^2\sigma_0 u_0 - C + \frac{\omega^2 a^2}{2} = 0,$$

$$4\pi\omega^2\sigma_2 u_2 + \frac{\omega^2 a^2}{2}\left(\tfrac{1}{3} - \mu^2\right) = 0,$$

and $\sigma_n u_n = 0$, for all values of n except 0 and 2.

Thus we have eliminated V without knowing what function it is, and obtained equations for determining u_0, u_1, u_2, &c., and therefore the equation to any stratum.

43. We shall now proceed to solve these equations, commencing with the last of them, viz.

$$\sigma_n u_n = 0 \ldots (1), \text{ except } n = 0 \text{ or } 2.$$

By (Art. 40) this equation is equivalent to

$$-\frac{u_n}{a}\int_0^a \rho' a'^2 da' + \frac{1}{(2n+1)\,a^{n+1}}\int_0^a \rho' \frac{d}{da'}(a'^{n+3}\,{}'u_n)\,da'$$
$$+\frac{a^n}{2n+1}\int_0^a \rho' \frac{d}{da'}\left(\frac{{}'u_n}{a'^{n-2}}\right) da' = 0.$$

Now, by Note, p. 24, if we multiply this equation by a^{n+1}, and differentiate it relatively to a, and then multiply it by $\frac{1}{a^{2n}}$, and differentiate it again relatively to a, we shall by this process get rid of the two last integral signs, and arrive at a differential equation which may be put in the form

$$\frac{d^2 u_n}{da^2} + A_1\frac{du_n}{da} + A_2 u_n = 0 \ldots\ldots (2),$$

A_1, A_2 being functions of a and ρ, which we have no occasion to determine. Now let v and v' be two quantities which satisfy the equations

$$\left.\begin{aligned} \sigma_n v - a^n &= 0 \\ \sigma_n v' - \frac{1}{a^{n+1}} &= 0 \end{aligned}\right\} \dots\dots\dots (3),$$

and let C and C' be two constants, then

$$u_n = Cv + C'v'$$

will be a solution of the differential equation (2); for, performing the operation σ_n on both sides of the equation,

$$u_n = Cv + C'v',$$

we have
$$\sigma_n u_n = \sigma_n (Cv + C'v')$$
$$= C\sigma_n v + C'\sigma_n v'$$

(evidently from the nature of the operation σ_n)

$$\text{or } \sigma_n u_n = Ca^n + C' \frac{1}{a^{n+1}}, \text{ by (3).}$$

Now if we get rid of the integral signs in this equation by the same process we have applied to the equation (1), the second side will evidently be made zero by the differentiations, and thus we shall arrive at the same differential equation as before; hence

$$u_n = Cv + C'v' *$$

is a solution of (2). Now this is evidently true whatever be the values of C and C'; hence this solution contains two arbitrary constants, and is therefore the *most general* solution that (2) admits of. Hence all values of u_n which satisfy (1), since they also satisfy (2), must be values of

* It is evident that v and v' are two different functions, otherwise the equations (3) would give

$$a^n = \frac{1}{a^{n+1}},$$

which is not the case. This remark is necessary, for if v and v' were not different functions then C and C' would add together, and therefore be equivalent to only one constant.

$Cv + C'v'$. To determine what values of $Cv + C'v'$ satisfy (1), substitute $Cv + C'v'$ for u_n in (1), and we find

$$C\sigma_n v + C'\sigma_n v' = 0,$$

$$\text{or} \quad Ca^n + C'\frac{1}{a^{n+1}} = 0, \text{ by (3).}$$

Now this equation ought to be true for all values of a; hence $C = 0$, and $C' = 0$; hence it is evident that only one value of $Cv + C'v'$, namely zero, satisfies (1); and hence it follows from the equation (1), that

$$u_n = 0.$$

Thus u_n is zero for all values of n except 0 and 2; this result produces a considerable simplification in the equation to the strata.

44. We shall next consider the equation involving u_2, which may be written thus,

$$\sigma_2 u_2 + \frac{a^2}{8\pi}(\tfrac{1}{3} - \mu^2) = 0.$$

In this equation we may conceive u_2, which is a function of μ, $\sqrt{1-\mu^2}$, $\cos\phi$, and $\sin\phi$, to be developed in a series of powers of $\frac{1}{3} - \mu^2$ and ϕ. Let $\gamma(\frac{1}{3} - \mu^2)^m \phi^n$ be any term in this development, γ being an unknown coefficient to be determined, then the corresponding term in the equation will be

$$(\sigma_2\gamma)(\tfrac{1}{3} - \mu^2)^m \phi^n,$$

except $m = 1$ and $n = 0$, in which case it will be

$$\left(\sigma_2\gamma + \frac{a^2}{8\pi}\right)(\tfrac{1}{3} - \mu^2).$$

Now $\frac{1}{3} - \mu^2$ and ϕ are arbitrary; hence we must put the coefficients of their several powers separately equal to zero; and hence for all terms, except that in which $m = 1$ and $n = 0$, we have

$\sigma_2\gamma = 0$; and therefore $\gamma = 0$, as before in (Art. 43);

and for the term in which $m = 1$ and $n = 0$,

$$\sigma_2 \gamma + \frac{a^2}{8\pi} = 0,$$

which equation will determine γ; hence we have

$$u_2 = \gamma \left(\tfrac{1}{3} - \mu^2\right),$$

where γ is given by the equation

$$\sigma_2 \gamma + \frac{a^2}{8\pi} = 0.$$

45. We cannot determine u_0 from the remaining equation in Art. 42, on account of the unknown constant C involved in it; but the value of u_0 follows from the assumption that we have made respecting a in (Art. 12); namely, that it equals the radius of the sphere of the same capacity as the stratum whose parameter it is: for this assumption gives

$$\frac{4\pi a^3}{3} = \text{volume included by stratum,}$$

$$= \int_0^{2\pi} \int_{-1}^{1} \int_0^{a(1+\omega^2 u)} r^2 dr\, d\mu\, d\phi$$

$$= \frac{a^3}{3} \int_0^{2\pi} \int_{-1}^{1} (1 + \omega^2 u)^3 d\mu\, d\phi$$

$$= \frac{a^3}{a} \int_0^{2\pi} \int_{-1}^{1} \left\{1 + 3\omega^2 (u_0 + u_1 + u_2 + \dots \&c.)\right\} d\mu\, d\phi$$

$$= \frac{4\pi a^3}{2} + 4\pi a^3 \omega^2 u_0, \qquad \text{by (Art. 22.)}$$

Hence $u_0 = 0$.

46. Thus we have determined the values of u_0, u_1, u_2, &c.; and it now only remains to substitute these values in the equation,

$$r = a \left\{1 + \omega^2 (u_0 + u_1 + u_2\ \&c.)\right\};$$

and we find that the equation to any stratum whose parameter is a, is

$$r = a\left\{1 + \omega^2\gamma\left(\tfrac{1}{3} - \mu^2\right)\right\},$$

$$= a\left\{1 + \omega^2\gamma\left(\tfrac{1}{3} - \cos^2\theta\right)\right\}.$$

Now the equation to a spheroid generated by an ellipse revolving about its minor axis, is

$$r = \frac{a\sqrt{1 - e^2}}{\sqrt{1 - e^2\cos^2\left(\frac{\pi}{2} - \theta\right)}};$$

a being the major axis, e the eccentricity, and θ the angle which r makes with the minor axis. Supposing e very small, this equation becomes

$$r = a\left\{1 - \frac{e^2}{2} + \frac{e^2}{2}\sin^2\theta\right\}$$

$$= a\left\{1 - \epsilon\cos^2\theta\right\},$$

ϵ being the ellipticity,

$$\text{for the ellipticity} = \frac{\text{major axis} - \text{minor axis}}{\text{major axis}}$$

$$= \frac{a - a\sqrt{1 - e^2}}{a} = \frac{e^2}{2}, \text{ nearly.}$$

This equation may evidently be made to coincide with the equation to the stratum, by putting

$$a = a\left(1 + \frac{\omega^2\gamma}{3}\right),$$

$$a\epsilon = a\omega^2;$$

and therefore $\epsilon = \dfrac{\omega^2\gamma}{1 + \dfrac{\omega^2\gamma}{3}} = \omega^2\gamma$, nearly.

Hence the strata are all spheroids of revolution about the polar axis; the ellipticity of any stratum being $\omega^2\gamma$, γ being got from the equation

$$\sigma_2\gamma + \frac{a^2}{8\pi} = 0;$$

or putting in this equation $\frac{\epsilon}{\omega^2}$ for γ, the equation for determining the ellipticity of any stratum, will be

$$\sigma_2\epsilon + \frac{a\omega^2}{8\pi} = 0,$$

or, by Art. 40,

$$-\frac{\epsilon}{a}\int_0^a \rho' a' da' + \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}(a'^5\epsilon')da'$$
$$+\frac{a^2}{5}\int_a^{a_1} \rho' \frac{d\epsilon'}{da'}da' + \frac{a^2\omega^2}{8\pi} = 0,$$

which equation will give ϵ when ρ' is known as a function of a'.

47. Thus we have arrived at the remarkable result that the mass must be arranged in strata of equal density, which are all spheroids of revolution about the axis of rotation, their ellipticities being connected by the equation just obtained.

It is evident that our investigation gives us no information respecting ρ; hence the law of density of the strata is quite arbitrary, and must be determined, if possible, by some independent method.

48. We shall presently shew that the results of our investigation may be compared with observation in a most satisfactory manner, without knowing any thing of the law of density of the strata. This is fortunate, as we have no means of determining this law, but must have recourse to an hypothesis which it must be confessed is rather empirical; but as the results it leads to may be made to agree well with

observation, we must look on it as probable. The hypothesis we allude to is this, that the variations of the pressure in the interior of the earth (supposed fluid and of the same chemical constitution all through,) are proportional, not to the variations of the density as in gases of uniform temperature, but to the variations of the square of the density; i.e. that instead of having $dp = k\,d\rho$, we have $dp = k\rho\,d\rho$. There is some slight reason *à priori* for assuming this formula, for it is evident that p ought to increase more rapidly with ρ in the fluid composing the earth, than it would do in gases, both on account of the incompressibility of that fluid, and the increase of temperature as we go towards the centre; and hence $k\rho\,d\rho$ will represent the variations of p better than $k\,d\rho$. But the chief reason for assuming this formula is, that it leads to correct results, and simplifies the equations we shall be concerned with, as will appear.

49. Assuming then this connection between the pressure and density, we may calculate the law of density from the equations

$$dp = \rho\,dV \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$V = \frac{4\pi}{a}\int_0^a \rho' a' d a' + 4\pi \int_a^{a_1} \rho' a' d a' \quad \ldots\ldots\ldots \quad (2);$$

which are got from (Art. 6) and (Art. 42); neglecting ω^2, for the centrifugal force will make a very little difference in the law of density, and it will be useless to be very accurate here, as we are proceeding on rather uncertain grounds.

We have, from (1),

$$\frac{dp}{da} = \rho\,\frac{dV}{da}$$

$$= -\frac{4\pi\rho}{a^2}\int_0^a \rho' a'^2 da', \text{ by (2)};$$

multiplying this by $\frac{a^2}{\rho}$, and differentiating relatively to a, we have

$$\frac{d}{da}\left(\frac{a^2}{\rho}\frac{dp}{da}\right) + 4\pi\rho a^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Substituting the assumed value of dp, viz. $dp = k\rho d\rho$, we have

$$\frac{d}{da}\left(a^2 k \frac{d\rho}{da}\right) + 4\pi\rho a^2 = 0,$$

which may be put in the form

$$\frac{d^2(\rho a)}{da^2} + \frac{4\pi}{k} \cdot \rho a = 0.$$

Therefore, putting $\frac{4\pi}{k} = q^2$,

$$\rho a = A \sin(qa + B),$$

$$\text{and } \rho = \frac{A \sin(qa + B)}{a},$$

A and B being arbitrary constants.

50. To determine B, let $a = 0$, then we have

$$\rho = \infty, \text{ unless } B = 0;$$

hence since ρ, as we may evidently assume, is not infinite at the centre, $B = 0$, and we have

$$\rho = \frac{A \sin qa}{a}.$$

To determine A and q, let ρ_1 be the density of the superficial parts of the earth, i.e. the value of ρ when $a = a_1$, and let D be the mean density of the earth; then

$$D = \frac{\text{mass of earth}}{\text{its volume}}$$

$$= \frac{4\pi \int_0^a \rho' a'^2 da'}{\frac{4\pi}{3} a^3}$$

$$= \frac{3A}{a^3} \int_0^{a_1} a' \sin q a' \, da' ;$$

putting $\frac{A \sin q a'}{a'}$ for ρ'

$$= \frac{3A}{q^2 a_1^3} (\sin q a_1 - q a_1 \cos q a_1),$$

also $\rho_1 = \frac{A}{a_1} \sin q a_1$;

hence $$\frac{\sin q a_1}{\sin q a_1 - q a_1 \cos q a_1} = \frac{3}{q^2 a_1^2} \frac{\rho_1}{D},$$

from which equation q may be found, and then we shall have A from the equation

$$A = \frac{a_1 \rho_1}{\sin q a_1}.$$

51. Observation shews that $\frac{\rho_1}{D}$ is about $\frac{5}{11}$, and on substituting this value of $\frac{\rho_1}{D}$ in the equation for determining q, we shall find by repeated trials (which is the only way we can solve it) that it admits of several solutions, of which one only leads to right results; it is this,

$$q a_1 = \text{about } \frac{5\pi}{6};$$

hence, substituting $\frac{1}{a_1} \frac{5\pi}{6}$ for q, we have

$$\rho = \frac{A \sin \left(\frac{5\pi}{6} \frac{a}{a_1} \right)}{a_1}.$$

To determine A, we shall put $a = a_1$ in this, which gives

$$\rho_1 = \frac{A}{a_1} \sin \frac{5\pi}{6}$$

$$= \frac{A}{a_1} \sin 30^0$$

$$= \frac{A}{a_1} \frac{1}{2};$$

and hence $A = 2\rho_1 a_1$, and we have

$$\rho = 2\rho_1 \frac{a}{a_1} \sin \left(\frac{5\pi}{6} \frac{a}{a_1}\right);$$

which, if our hypotheses be correct, expresses the density of any stratum in terms of the parameter of that stratum, and the superficial density.

52. The method by which we have arrived at this formula for the density is not very satisfactory, and we shall therefore consider it as empirical; we observe that it gives a density which increases as we go towards the centre, but does not become infinite there; this is most probably the case; it also makes the pressure vary more rapidly as we approach the centre than it would do if the earth were gaseous and of uniform temperature; this is also most probably the case; and it gives the mean density of the earth its proper value: we shall prove presently that it gives the true value of the earth's ellipticity, and also the true value of the coefficient of precession; hence, on the whole, we may assume it with some probability as the law of density.

53. Hence, finally, it follows from the *hypothesis of the earth's original fluidity;*

(1) *That the earth ought to consist of equidense strata, all spheroids of revolution about the axis of rotation.*

(2) *That if* ϵ *be the ellipticity of any of these strata it satisfies the equation*

$$-\frac{\epsilon}{a}\int_0^a \rho' a'^2 da' + \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}(\epsilon' a'^5)\, da'$$

$$+\frac{a^2}{5}\int_a^{a_1} \rho' \frac{d\epsilon'}{da'}\, da' + \frac{a^2\omega^2}{8\pi} = 0.$$

(3) *That we may assume with probability the law of density to be*

$$\rho = 2\rho_1 \frac{a}{a_1} \sin\left(\frac{5\pi}{6}\frac{a}{a_1}\right).$$

CHAPTER IV.

METHODS OF COMPARING THE RESULTS JUST ARRIVED AT WITH OBSERVATION.

54. In order to test the correctness of the conclusions we have just arrived at, we shall now deduce from them results of a more practical character, which shall admit of direct comparison with observation: the first result we shall deduce is this;

If s be the length of a meridian arc, measured from the pole to any place whose colatitude is c, then

$$s = a\left\{\left(1 - \frac{\epsilon_{,}}{2}\right) c + \frac{3\epsilon_{,}}{4} \sin 2c\right\},$$

a and $\epsilon_{,}$ being the major axis and ellipticity of the earth.

For, by Art. 48, the equation to any meridian is

$$r = a\left(1 - \epsilon_{,} \cos^2 \theta\right)$$

$$= a\left(1 - \frac{\epsilon_{,}}{2} - \frac{\epsilon_{,}}{2} \cos 2\theta\right) \ldots\ldots\ldots (1).$$

$$\text{Now } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= r, \text{ neglecting squares \&c. of } \epsilon,$$

$$= a\left\{1 - \frac{\epsilon_{,}}{2} - \frac{\epsilon_{,}}{2} \cos 2\theta\right\} \text{ by (1)},$$

therefore, integrating,

$$s = a\left\{\left(1 - \frac{\epsilon_{,}}{2}\right) \theta - \frac{\epsilon_{,}}{4} \sin 2\theta\right\} \ldots\ldots\ldots\ldots (2).$$

Adding no constant because s evidently $= 0$ when $\theta = 0$.

Now c (the colatitude) is the angle made by the normal with the polar axis; hence $(\theta - c)$ is the angle made by the normal and radius vector, and hence

$$\tan(\theta - c) = \frac{1}{r}\frac{dr}{d\theta};$$

or, neglecting the squares &c. of small quantities,

$$\theta - c = \epsilon \sin 2\theta, \text{ by } (1),$$

therefore

$$\begin{aligned}\theta &= c + \epsilon_{,} \sin 2\theta \\ &= c + \epsilon_{,} \sin 2(c + \epsilon \sin 2\theta) \\ &= c + \epsilon_{,} \sin 2c \text{ nearly};\end{aligned}$$

hence, substituting in (2), we have

$$\begin{aligned}s &= a\left\{c + \epsilon_{,} \sin 2c - \frac{\epsilon_{,}}{2} c - \frac{\epsilon_{,}}{4} \sin 2c\right\}, \\ &= a\left\{\left(1 - \frac{\epsilon_{,}}{2}\right) c + \frac{3\epsilon_{,}}{4} \sin 2c\right\}.\end{aligned}$$

55. To shew how this result may be compared with observation, let s' and c' be the values of s and c corresponding to another place near the former, and on the same meridian, then

$$s' = a\left\{\left(1 - \frac{\epsilon_{,}}{2}\right) c' + \frac{3\epsilon_{,}}{4} \sin 2c'\right\};$$

and therefore

$$\begin{aligned}s' - s &= a\left\{\left(1 - \frac{\epsilon_{,}}{2}\right)(c' - c) + \frac{3\epsilon_{,}}{4}(\sin 2c' - \sin 2c)\right\}, \\ &= a\left\{\left(1 - \frac{\epsilon_{,}}{2}\right)(c' - c) + \frac{3\epsilon_{,}}{2} \cos(c' + c) \sin(c' - c)\right\}.\end{aligned}$$

Now in this equation $s' - s$, being the distance between two places near each other, may be determined by the usual method of triangulation; cos $(c + c')$ may be found by any of the ordinary methods of determining the latitude of places, without aiming at any great accuracy, since it is multiplied by the small quantity ϵ; $c' - c$, not being multiplied by a small quantity, must be determined more accurately by observing the meridian zenith distances of the same star at the two places, and taking the difference which will evidently be equal to $c' - c$; thus we may put our equation in the form

$$A = a\,(B + C\epsilon_{,}),$$

when A, B, C are known quantities got by observation.

In the same way, by observations at other places, we may obtain any number of similar equations; suppose them to be

$$A' = a\,(B' + C'\epsilon_{,}),$$
$$A'' = a\,(B'' + C''\epsilon_{,}),$$
&c....... &c.......

From any two of these equations we may determine a and $\epsilon_{,}$; and the values of a and ϵ so determined* ought to satisfy all the other equations; hence, if we find that all these equations are satisfied by the same values of a and $\epsilon_{,}$, it is evident that our result agrees with observation.

56. Now a number of meridian arcs have been measured, and a system of equations similar to the above have been formed, and it is found that the values

$$a = 3962.82 \text{ miles}, \quad \epsilon_{,} = \frac{1}{306},$$

satisfy them all to a remarkable degree of accuracy, allowing for certain small errors which may be easily accounted for; and which, even considering them in the most unfavourable point of view, are very much smaller than $\epsilon_{,}$, which is itself

* Or rather, the values of a and $\epsilon_{,}$ got from all the equations by the method of least squares.

a very small quantity; and indeed if we bear in mind the delicacy and number of the observations requisite in order to form the above equations, the smallness of the errors is most remarkable.

57. On the whole we are justified in concluding from observation, that the equation to any meridian and therefore to the earth's surface is very nearly this, viz.

$$r = a\left\{1 - \epsilon_{,} \cos^2\theta\right\},$$

where $a = 3962.82$ miles,

$$\text{and } \epsilon_{,} = \frac{1}{306}.$$

Hence the hypothesis of the earth's original fluidity leads to a very peculiar result, which is capable of varied and extensive comparison with observation, and which agrees with it in a remarkable manner; from which we must conclude that this hypothesis is most probably true.

58. The second result we shall deduce from our theory is this;

If g be the force of gravity at any place whose colatitude is c, then

$$g = G\left\{1 + \left(\frac{5m}{2} - \epsilon_1\right)\cos^2 c\right\}.$$

Where G is a constant, namely, the value of g at the equator, and m the ratio of the centrifugal force to gravity at the equator*.

* To determine m, we observe that

$$G = 32.2 + \text{a small quantity},$$

hence

$$m = \frac{\omega^2 a}{32.2 + \text{a small quantity}}$$

$$= \frac{\omega^2 a}{32.2}, \text{ very nearly, since } \omega^2 \text{ is small.}$$

In putting 32.2 for G we have assumed a foot to be the unit of length, and a second the unit of time; hence we must express a and ω^2 in terms of these units, and therefore we have

$$a = 3962 \text{ miles, nearly,}$$

$$= 3962 \times 5280 \text{ feet,}$$

and

To prove this, we observe that $-g$ is the resultant of the forces which act on a particle at the surface, which forces are, by (6),

$$\frac{dV}{dx} + \omega^2 x,\ \frac{dV}{dy} + \omega^2 y,\ \frac{dV}{dz},$$

and therefore g must balance these forces; hence, by the principle of virtual velocities, if dr be any variation of r, and dx, dy, dz, its resolved parts along the axes of co-ordinates, and ψ the angle which r makes with the direction of g, we shall have

$$0 = g\,dr\cos\psi + \left(\frac{dV}{dx} + \omega^2 x\right) dx + \left(\frac{dV}{dy} + \omega^2 y\right) dy + \frac{dV}{dz}\,dz,$$

which gives

$$g = -\frac{1}{\cos\psi}\,\frac{d\left\{V + \frac{\omega^2}{2}(x^2 + y^2)\right\}}{dr} \quad \text{.........} \ (1).$$

Now g acts in the normal (by the principles of Hydrostatics), and the normal evidently makes a very small angle with the radius vector; hence ψ is very small, and therefore, since $\cos\psi = 1 - \frac{\psi^2}{2} + \&c.$, we may, neglecting squares and higher powers of small quantities, put $\cos\psi = 1$; moreover we have, by (6) and (42),

$$V + \frac{\omega^2}{2}(x^2 + y^2) = C,$$

and

$$C = \frac{4\pi}{a}\int_0^a \rho' a'^2 da' + 4\pi \int_a^{a_1} \rho' a'\, da' + \frac{\omega^2 a^2}{3};$$

$$\text{and } \omega = \frac{2\pi}{24 \text{ hours}} = \frac{2\pi}{24 \times 60 \times 60}.$$

Making these substitutions, we shall find

$$m = \text{about } \frac{1}{289}.$$

hence, observing that a is a function of r in virtue of the equation

$$r = a\left\{1 + \epsilon\left(\tfrac{1}{3} - \mu^2\right)\right\},$$

we have, substituting in (1),

$$g = -\frac{dC}{dr}$$

$$= -\frac{dC}{dr}\frac{da}{dr}$$

$$= \left(\frac{4\pi}{a^2}\int_0^a \rho' a'^2 da' - \frac{2\omega^2 a}{3}\right)\left\{1 - \frac{d(a\epsilon)}{da}\left(\tfrac{1}{3} - \mu^2\right)\right\},$$

which, neglecting the product of small quantities, may evidently be put in the form

$$g = G\left\{1 + \frac{d(a\epsilon)}{da}\mu^2\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where

$$G = \frac{4\pi}{a^2}\int_0^a \rho' a'^2 da' + \text{small quantities}.$$

Since $g = G$ when $\mu = 0$, it is evident that G is the force of gravity at the equator.

(Of course, in all our formulæ, $a_{,}$ is supposed to be put for a after all differentiations and integrations have been performed, since the particle on which g is the force is supposed to be at the surface.)

Now by (53),

$$-\frac{\epsilon}{a}\int_0^a \rho' a'^2 da' + \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}(\epsilon' a'^5)\, da'$$

$$+ \frac{a^2}{5}\int_a^{a_1} \rho' \frac{d\epsilon'}{da'} da' + \frac{a^2\omega^2}{8\pi} = 0.$$

Multiplying this equation by a^3, and differentiating relatively to a, we have

$$-\frac{d(\epsilon a^2)}{da}\int_0^a \rho' a'^2 da' + a^4\int_a^{a_1} \rho' \frac{d\epsilon'}{da'} da' + \frac{5\omega^2 a^4}{8\pi} = 0,$$

or, observing that $a_{\prime}$ is to be put for a,

$$\frac{d(\epsilon a^2)}{da} = \frac{5\omega^2 a^4}{8\pi \int_0^{a_1} \rho' a'^2 da'}$$

$$= \frac{5\omega^2 a^2}{2G},$$

since $G = \dfrac{4\pi}{a^2}\displaystyle\int_0^a \rho' a'^2 da'$ + small quantities,

$$= \frac{5ma}{2},$$

since $m = \dfrac{\text{centrifugal force}}{\text{gravity at equator}} = \dfrac{\omega^2 a}{G}$;

hence $\dfrac{d(\epsilon a)}{da} = \dfrac{1}{a}\dfrac{d(\epsilon a^2)}{da} - \epsilon_{\prime}$

$$= \frac{5m}{2} - \epsilon_{\prime};$$

and hence equation (1) becomes

$$g = G\left\{1 + \left(\frac{5m}{2} - \epsilon_{\prime}\right)\cos^2\theta\right\}$$

$$= G\left\{1 + \left(\frac{5m}{2} - \epsilon_{\prime}\right)\cos^2 c\right\},$$

since $\theta = c + \epsilon \sin 2c$, by (64).

59. To shew how to test this result by observation, we observe that if p be the length of the seconds pendulum, then since $1 = 2\pi\sqrt{\dfrac{p}{g}}$, we have $p = \dfrac{g}{4\pi^2}$

$$= P\left\{1 + \left(\frac{5m}{2} - \epsilon_{\prime}\right)\cos^2 c\right\},$$

when $P = \dfrac{G}{4\pi^2}$ = value of p at the equator.

Now p may be determined by observation at any place; hence, by observations at different places, we may find a system of equations such as before; viz.

$$A = P\left\{1 + \left(\frac{5m}{2} - \epsilon_{,}\right) B\right\},$$

$$A' = P\left\{1 + \left(\frac{5m}{2} - \epsilon_{,}\right) B'\right\},$$

$$A'' = P\left\{1 + \left(\frac{5m}{2} - \epsilon_{,}\right) B''\right\},$$

&c. &c.

when A, B, A', B'', &c....are quantities got from observation.

Now it is found that the values

$$P = 39.01228 \text{ inches, and } \frac{5m}{2} - \epsilon_{,} = .005321,$$

satisfy all these equations, not so exactly as before, but yet with remarkable accuracy, considering the small quantities we are engaged with; hence this result is another proof of the Earth's original fluidity.

60. The comparison of this result with the former is a strong *additional* proof; for the former result gives

$$\epsilon_{,} = \frac{1}{306},$$

and since we know that $m = \frac{1}{289}$, the present result gives

$$\epsilon_{,} = \frac{1}{306}, \text{ nearly;}$$

this is a very remarkable coincidence, and must be considered as a decided proof of the correctness of our hypothesis.

61. If G' be the value of g at the pole,

$$G' = G\left\{1 + \left(\frac{5m}{2} - \epsilon_{,}\right)\right\},$$

$$\text{and } \frac{G'-G}{G} = \frac{5m}{2} - \epsilon_{\prime}.$$

This result is Clairaut's theorem.

62. We may determine the Earth's ellipticity by means of the law of density assumed in Chap. III.

The equation for determining ϵ is

$$-\frac{\epsilon}{a}\int_0^a \rho' a'^2 da' + \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}(\epsilon' a'^2)\, da'$$
$$+ \frac{a^2}{5}\int_a^{a_1} \rho' \frac{d\epsilon'}{da'} da' + \frac{\omega^2 a^2}{8\pi} = 0.$$

Integrating the second and third terms by parts, we have

$$(1) \ldots 0 = -\frac{\epsilon}{a}\int_0^a \rho' a'^2 da' - \frac{1}{5a^3}\int_0^a \frac{d\rho'}{da'} a'^5 \epsilon' da'$$
$$-\frac{a^2}{5}\int_a^{a_1} \frac{d\rho'}{da'}\epsilon' da' + \frac{\rho_{\prime}\epsilon_{\prime} a^2}{5} + \frac{\omega^2 a^2}{8\pi}.$$

Now putting $\rho = \dfrac{A \sin qa}{a}$, we have

$$\int_0^a \rho' a'^2 da' = \frac{A}{q^2}(\sin qa - qa\cos qa)$$

$$\frac{d\rho'}{da'} = -\frac{A}{a'^2}(\sin qa' - qa' \cos qa');$$

hence, if we assume

$$\epsilon(\sin qa - qa\cos qa) = \eta,$$

(1) will become

$$-\frac{A}{q^2}\frac{\eta}{a} + \frac{A}{5a^3}\int_0^a \eta' a'^3 da' + \frac{Aa^2}{5}\int_a^{a_1}\frac{\eta_{\prime}}{a'^2} da' + \left(\frac{\rho_{\prime}\epsilon_{\prime}}{5} + \frac{\omega^2}{8\pi}\right)a^2 = 0;$$

dividing this by a^2, and differentiating relatively to a, we have

$$-\frac{A}{q^2}\frac{d}{da}\left(\frac{\eta}{a^3}\right) - \frac{A}{a^6}\int_0^a \eta' a'^3 da' = 0;$$

or multiplying by a^6, and differentiating again,

$$\frac{d}{da}\left\{a^6\frac{d}{da}\left(\frac{\eta}{a^3}\right)\right\}+q^2\eta a^3=0\ \ldots\ldots\ldots\ (2),$$

$$\text{or}\quad \frac{d^2\eta}{da^2}+\left(q^2-\frac{6}{a^2}\right)\eta=0.$$

63. To solve this differential equation we shall assume ζ' such a function of a' that

$$\eta=\frac{1}{a^2}\int_0^a a'\int_0^a a'\zeta'\,da'^2,$$

therefore

$$a^6\frac{d}{da}\left(\frac{\eta}{a^2}\right)=a^2\int_0^a a'\zeta'\,da'-5\int_0^a a'\int_0^a a'\zeta'\,da'^2,$$

$$\text{and}\quad \frac{d}{da}\left\{a^6\frac{d}{da}\left(\frac{\eta}{a^3}\right)\right\}=a^3\zeta-3a\int_0^a a'\zeta'd\,a',$$

Hence, substituting in the equation (2), we have

$$a^3\zeta-3a\int_0^a a'\zeta'da'+q^2a\int_0^a a'\int_0^a a'\zeta'\,da'^2;$$

or dividing by a and differentiating,

$$a^2\frac{d\zeta}{da}-a\zeta+q^2a\int_0^a a'\zeta'\,da';$$

or again dividing by a and differentiating,

$$\frac{d^2\zeta}{da^2}+q^2\zeta=0;$$

hence $\zeta=C\sin(qa+C')$, C and C' being arbitrary constants, and

$$\int_0^a a'\zeta'\,da'=\frac{C}{q^2}\{\sin(qa+C')-qa\cos(qa+C')\},$$

and

$$\int_0^a a'\int_0^a a'\zeta'\,da'=\frac{C}{q^4}\{3[\sin(qa+C')-qa\cos(qa+C')]$$
$$-q^2a^2\sin(qa+C')\}.$$

Hence we shall have, putting C instead of $\frac{C}{q^6}$,

$$\eta = C\left\{\frac{3}{q^2a^2}\left[\sin(qa+C') - qa\cos(qa+C')\right] - \sin(qa+C')\right\}.$$

64. We might determine the constants C and C' by substituting this value of η in (1), Art. 62, but the following method will be more simple: in the first place we may see, *à priori*, that C' must $= 0$, for otherwise we should have η, and therefore ϵ very large when a is very small, contrary to our assumption of the nearly spherical form of the strata; hence

$$\eta = C\left\{\frac{3}{q^2a^2}(\sin qa - qa\cos qa) - \sin qa\right\} \ldots\ldots (2),$$

and therefore, since $\eta = \epsilon(\sin qa - qa\cos qa)$,

$$\epsilon = C\left\{\frac{3}{q^2a^2} - \frac{\sin qa}{\sin qa - qa\cos qa}\right\} \ldots\ldots\ldots\ldots\ldots (3).$$

We shall determine C by means of the value of $\frac{d(\epsilon a^2)}{da}$, got in p. 51.

Multiplying (2) by a^2, and differentiating relatively to a, we have

$$\frac{d(\eta a^2)}{da} = Ca(\sin qa - qa\cos qa),$$

also doing the same to the equation

$$\eta = \epsilon(\sin qa - qa\cos qa),$$

we have

$$\frac{d(\eta a^2)}{da} = \frac{d(\epsilon a^2)}{da}(\sin qa - qa\cos qa) + \epsilon q^2a^3\sin qa.$$

Equating these values of $\frac{d(\eta a^2)}{da}$, we have

$$C = \frac{1}{a}\frac{d(\epsilon a^2)}{da} + \epsilon q^2a^2\frac{\sin qa}{\sin qa - qa\cos qa}.$$

Hence putting $a = a_{,}$, and $\dfrac{d(\epsilon a^2)}{da} = \dfrac{5ma_{,}}{2}$, see p. 51,

$$C = \frac{5m}{2} + \frac{3\epsilon_{,}\rho_{,}}{D},$$

also putting $a = a_{,}$ in (3), we have by Art. 50,

$$\epsilon_{,} = C\frac{3}{q^2 a_{,}^2}\left(1 - \frac{\rho_{,}}{D}\right),$$

from which two equations we get

$$\epsilon_{,} = \frac{\dfrac{5m}{2}}{\dfrac{q^2 a_{,}^2}{3\left(1 - \dfrac{\rho_{,}}{D}\right)} - 3\dfrac{\rho_{,}}{D}}.$$

Now $m = \dfrac{1}{289}$, $qa_{,} = \dfrac{5\pi}{6}$, $\dfrac{\rho_{,}}{D} = \dfrac{5}{11}$,

substituting these values and reducing, we find

$$\epsilon_{,} = \text{about } \frac{1}{306}.$$

This result agrees with observation, but the agreement is not of much value on account of the assumption of the law of density.

65. The effect which the attraction of the Earth has on the Moon's motion, is usually brought forward as another means whereby we may test the correctness of the hypothesis of original fluidity; and the agreement between theory and observation in this particular is considered to afford *additional* evidence of the truth of the hypothesis. We shall attempt however to prove that this is not the case. To do this, we shall shew that if the equation to the Earth's surface be known, and also the law of variation of the force of gravity, then the effect of the Earth's attraction on the Moon follows as a necessary consequence, independently of any theory except that of universal gravitation.

66. It is evident from the smallness of the variations of the force of gravity, that the Earth must consist of nearly spherical strata; hence all the results we have already obtained, so far as they depend on the nearly spherical form of the strata, will be true whether the hypothesis of original fluidity be correct or not.

Hence, as before in p. 49, we shall have

$$g = -\frac{d}{dr}\left\{V + \frac{\omega^2}{2} r^2 (1-\mu^2)\right\}$$

$$= -\frac{dV}{dr} - \omega^2 r(1-\mu^2)$$

$$= -\frac{dV}{dr} - \omega^2 a_{,}(1-\mu^2),$$

putting $a_{,}$ for r in the small term.

*Now the expression for V in Art. 40. may evidently be put in the form

$$V = \frac{4\pi}{r}\int_0^a \rho' a'^2 da' + 4\pi \int_a^{a_1} \rho' a' da'$$

$$+ 4\pi\omega^2 \left\{ \begin{array}{l} \text{a series whose general term is} \\ \dfrac{1}{(2n+1)\, a^{n+1}} \displaystyle\int_0^a \rho' \frac{d}{da'}({}'u_n a'^{n+3})\, da' \\ \qquad + \dfrac{a^n}{2n+1} \displaystyle\int_a^{a_1} \rho' \frac{d}{da'}\left(\frac{{}'u_n}{a'^{n+2}}\right) da' \end{array} \right\};$$

hence, differentiating relatively to r, and observing that a is a function of r in virtue of the equation $r = a(1+\omega^2 u)$, and supposing that $a_{,}$ is put for a after the differentiations, we have

$$\frac{dV}{dr} = -\frac{4\pi}{r_{,}^2}\int_0^{a_1} \rho' a'^2 da' + \left(\frac{4\pi}{r'}\rho_{,} a_{,}^2 - 4\pi\rho a_{,}\right)\frac{da}{dr},$$

* For the part of V which is multiplied by $4\pi \int_0^a \rho' a'^2 da'$ is evidently

$$\frac{1}{a}\{1 - \omega^2(u_0 + u_1 + u_2, \text{ \&c.})\}, \text{ which} = \frac{1}{r}.$$

$$+ 4\pi\omega^2 \left\{ \begin{array}{l} \text{a series whose general term is} \\ -\dfrac{n+1}{(2n+1)a_{\prime}^{n+2}} \displaystyle\int_0^{a_1} \rho' \frac{d}{da'}('u_n a'^{n+3}) da' + \rho_{\prime} a_{\prime} u_n \end{array} \right\} \frac{da}{dr}.$$

Now since $r = a(1 + \omega^2 u)$,

$$\text{and therefore } \frac{da}{dr} = 1 - \omega^2 \frac{d(au)}{da},$$

we have, evidently, neglecting ω^4 &c.,

$$\left(\frac{4\pi}{r_{\prime}} \rho_{\prime} a_{\prime} 2 - 4\pi\rho a_{\prime}\right) \frac{da}{dr} = -4\pi\omega^2 \rho_{\prime} a_{\prime} u$$

$$= -4\pi\omega^2 \rho_{\prime} a_{\prime} (u_0 + u_1 + \&\text{c.}),$$

and therefore, neglecting ω^4 &c.,

$$\frac{dV}{dr} = -\frac{4\pi}{r_{\prime}^2} \int_0^{a_1} \rho' a'^2 da'$$

$$+ 4\pi\omega^2 \left\{ \begin{array}{l} \text{a series whose general term is} \\ -\dfrac{n+1}{(2n+1)a_{\prime}^{n+2}} \displaystyle\int_0^{a_1} \rho_{\prime} \frac{d}{da'}('u_n a'^{n+3}) da' \end{array} \right\};$$

hence if, for brevity, we put

$$\frac{1}{(2n+1)} \int_0^{a_1} \rho' \frac{d}{da'}('u_n a'^{n+3}) da' = Z_n,$$

we shall have

$$g = \frac{4\pi}{r_1^2} \int_0^{a_1} \rho' a'^2 da' + \frac{4\pi\omega^2}{a_1^2} \left\{ Z_0 + \frac{2}{a_1} Z_1 + \frac{3}{a_1^2} Z_2 + \&\text{c.} \right\}$$

$$- \omega^2 a_1 (1 - \mu^2);$$

and hence

$$g - \frac{4\pi}{r_1^2} \int_0^{a_1} \rho' a'^2 da' + \omega^2 a_1 (1 - \mu^2)$$

$$= \frac{4\pi\omega^2}{a_1^2} \left\{ Z_0 + \frac{2}{a_1} Z_1 + \frac{3}{a_1^2} Z_2 + \&\text{c.} \right\}.$$

Now if we suppose r_1 and g known, it is evident that the first member of this equation will be known, and may therefore be supposed to be expanded in a series of known Laplace's coefficients; and hence, since the Laplace's coefficients of different orders on each side of the equation must be separately equal, by Art. 36, the values of Z_0 Z_1 Z_2 &c. will be known.

*Now it is evident immediately, from Arts. 14. and 40, that the value of V for any external point is

$$V = \frac{4\pi}{r}\int_0^{a_1} \rho' a'^2 da' + 4\pi\omega^2 \left\{\frac{1}{r} Z_0 + \frac{1}{r^2} Z_1 + \frac{1}{r^3} Z_2 + \&c.\right\};$$

hence, since Z_0 Z_1 Z_2 &c. are known, as we have just proved, it is evident that the value of V for any external point is also known.

67. Hence, if we know the form of the Earth's surface, and the law of the variation of gravity, we shall know the value of V for any external point, and therefore be able to determine the attractions of the Earth on that point without making use of the hypothesis of original fluidity.

Hence it follows that if the form of the Earth's surface and the law of variation of the force of gravity, calculated on the hypothesis of original fluidity, agree with observation, then the effect of the Earth's attraction on the motion of any external body, such as the Moon, calculated on the same hypothesis, must also agree with observation, whether that hypothesis be true or not; and hence we conclude, that the motion of the Moon does not afford *additional* evidence of the Earth's original fluidity.

* For the only difference made in the reasoning in Art. 40, by using the expression for V given in Art. 14 instead of that given in Art. 15, will be simply this, that we shall have to consider $\frac{a'}{r}$ instead of $\frac{a'}{a}$; also, since the attracted point is external, and therefore r always greater than a', it is evident that k will be always unity.

In the next chapter we shall determine the equations of motion of a rigid body round its centre of gravity, and thence deduce the Earth's motion round its centre of gravity. We shall find that the result affords a confirmation of the law of density assumed in Chapter III., and also of the hypothesis of original fluidity.

CHAPTER V.

EQUATIONS OF MOTION OF A RIGID BODY ROUND ITS CENTRE OF GRAVITY.

68. We know from the principles of Dynamics, that a rigid body acted on by any forces moves relatively to its centre of gravity, in the same manner as if that point became fixed, all other dynamical circumstances remaining unaltered; hence, whenever we wish to investigate the motion of a body relatively to its centre of gravity, we may consider that point as fixed, and this will render the investigation simpler.

Suppose, then, that we have a body whose centre of gravity is fixed, acted on by any forces; let δm be any element of it, $x\ y\ z$ the co-ordinates of δm at the time t referred to any arbitrary rectangular axes fixed in space, and originating in the centre of gravity; let $L\ M\ N$ be the moments of the impressed forces round the axes of $x\ y\ z$ respectively, then we have, by the principles of Dynamics,

$$\left.\begin{aligned} \Sigma\,\delta m\left(\frac{d^2y}{dt^2}x - \frac{d^2x}{dt^2}y\right) &= N \\ \Sigma\,\delta m\left(\frac{d^2x}{dt^2}z - \frac{d^2z}{dt^2}x\right) &= M \\ \Sigma\,\delta m\left(\frac{d^2z}{dt^2}y - \frac{d^2y}{dt^2}z\right) &= L \end{aligned}\right\}\ \ldots\ldots\ldots\ (A).$$

69. In order to perform the integrations denoted by Σ in these equations, we shall introduce new variables instead of $x\ y\ z$, which shall have reference merely to the

position and motion of the whole body, and not to any particular particle of it. To do this,

Let $x' y' z'$ be the co-ordinates of δm referred to any arbitrary rectangular axes fixed in the body, then we have

$$x = x' \cos(x'x) + y' \cos(y'x) + z' \cos(z'x);$$

differentiating this relatively to t, and observing that $x' y' z'$ do not vary with t, we have

$$\frac{dx}{dt} = x' \frac{d\cos(x'x)}{dt} + y' \frac{d\cos(y'x)}{dt} + z' \frac{d\cos(z'x)}{dt}.$$

Now the axes of x' y' z' are perfectly arbitrary; we may therefore suppose them so chosen that they shall coincide with the axes of x y z at any instant we please. Suppose therefore that this coincidence takes place at the time t, then we shall have $x' = x$, $y' = y$, $z' = z$; and if, for brevity, we denote the values of $\frac{d\cos(x'x)}{dt}$, $\frac{d\cos(y'x)}{dt}$, $\frac{d\cos(z'x)}{dt}$, at the instant of coincidence by λ λ' λ'' respectively, our equation becomes

$$\frac{dx}{dt} = \lambda x + \lambda' y + \lambda'' z;$$

λ λ' λ'' are evidently variables which have reference to the position and motion of the whole body, and not to any particular particle of it, for they depend simply on the angles which the two systems of co-ordinate axes make with each other at any time, or rather upon the rate at which these angles are varying at the instant of coincidence of the axes.

In the same way we may prove that

$$\frac{dy}{dt} = \mu y + \mu' z + \mu'' x,$$

$$\frac{dz}{dt} = \nu z + \nu' x + \nu'' y;$$

where μ μ' μ'', ν ν' ν'' are quantities similar to λ λ' λ''.

Now since δm is rigidly connected with the origin, we have

$$x^2 + y^2 + z^2 = \text{constant},$$

$$\text{and therefore } x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0;$$

substituting in this equation the values of $\frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}$ just found, we have

$$\lambda x^2 + \mu y^2 + \nu z^2 + (\lambda' + \mu'')xy + (\mu' + \nu'')yz + (\nu' + \lambda'')zx = 0;$$

hence, since $x\ y\ z$ are arbitrary, we have

$$\lambda = 0 \quad \mu = 0 \quad \nu = 0,$$

$$\lambda' + \mu'' = 0 \quad \mu' + \nu'' = 0 \quad \nu' + \lambda'' = 0.$$

Hence the values of $\frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}$ become

$$\frac{dx}{dt} = \lambda'' z - \mu'' y,$$

$$\frac{dy}{dt} = \mu'' x - \nu'' z,$$

$$\frac{dz}{dt} = \nu'' y - \lambda'' x.$$

To conform to a common and convenient notation, we shall put $\omega_1\ \omega_2\ \omega_3$ instead $\nu''\ \lambda''\ \mu''$ respectively, and write these equations thus,

$$\left.\begin{aligned} \frac{dx}{dt} &= \omega_2 z - \omega_3 y \\ \frac{dy}{dt} &= \omega_3 x - \omega_1 z \\ \frac{dz}{dt} &= \omega_1 y - \omega_2 x \end{aligned}\right\} \dots\dots\dots (B);$$

we shall presently determine what $\omega_1\ \omega_2\ \omega_3$ are.

These equations express the relations which exist between the velocities of any element of the body and its co-

ordinates at the time t, in consequence of the rigidity of the body. The substitution of these values of $\frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}$ in the equations (A) will be very advantageous, since $\omega_1\ \omega_2\ \omega_3$ are independent of $x\ y\ z$, and may therefore be brought outside the integral sign Σ. To perform this substitution we have from the equations (B), differentiating and in the result putting for $\frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}$ their values given by the equations (B),

$$\frac{d^2x}{dt^2} = \frac{d\omega_2}{dt}z - \frac{d\omega_3}{dt}y + \omega_2(\omega_1 y - \omega_2 x) - \omega_3(\omega_3 x - \omega_1 z)$$

$$\frac{d^2y}{dt^2} = \frac{d\omega_3}{dt}x - \frac{d\omega_1}{dt}z + \omega_3(\omega_2 z - \omega_3 y) - \omega_1(\omega_1 y - \omega_1 x),$$

$$\frac{d^2z}{dt^2} = \frac{d\omega_1}{dt}y - \frac{d\omega_2}{dt}x + \omega_1(\omega_3 x - \omega_1 z) - \omega_2(\omega_2 z - \omega_3 y).$$

Now the axes of $x\ y\ z$ are perfectly arbitrary, we may therefore choose them so that the principal axes of the body shall coincide with them at the time t, and therefore we shall have at the time t,

$$\Sigma\,\delta mxy = 0 \quad \Sigma\,\delta myz = 0 \quad \Sigma\,\delta mzx = 0,$$

and hence, from the equations just obtained, we have at the time t,

$$\Sigma\,\delta m\frac{d^2x}{dt^2}y = -\frac{d\omega_3}{dt}\Sigma\,\delta my^2 + \omega_2\omega_1\Sigma\,\delta my^2,$$

$$\Sigma\,\delta m\frac{d^2y}{dt}x = \frac{d\omega_3}{dt}\Sigma\,\delta mx^2 + \omega_1\omega_2\Sigma\,\delta mx^2;$$

and hence the first of the equations (A) will become

$$\frac{d\omega_3}{dt}\Sigma\,\delta m(x^2+y^2) + \omega_1\omega_2\Sigma\delta m(x^2-y^2) = N.$$

Now the principal axes coincide with the axes $x\ y\ z$ at the time t, hence if $A\ B\ C$ be the principal moments of inertia of the body, we shall have

$$\Sigma\,\delta m\,(x^2 + y^2) = C,$$

$$\Sigma\,\delta m\,(x^2 - y^2) = \Sigma\,\delta m\,(x^2 + z^2) - \Sigma\,\delta m\,(y^2 + z)$$
$$= B - A,$$

and hence $A\,\dfrac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 = N.$

We may transform the other two of the equations (A) in exactly the same manner, hence, instead of the equations (A), we have the following three equations, in which the integral sign Σ is got rid of, viz.

$$\left.\begin{array}{l} A\,\dfrac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 = L \\[2ex] B\,\dfrac{d\omega_2}{dt} + (A - C)\,\omega_3\omega_1 = M \\[2ex] C\,\dfrac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 = N \end{array}\right\}\,\cdots\cdots\cdots(A').$$

70. It remains to determine what $\omega_1\ \omega_2\ \omega_3$ are; the equations (B) will enable us to do this immediately, for from these equations we find (putting $z = 0$) that the velocities parallel to the axes of x and y of any particle in the plane of xy are

$$\frac{dx}{dt} = -\,\omega_3 y,$$

$$\frac{dy}{dt} = \quad \omega_3 x;$$

and hence the whole resolved velocity in the plane of xy of that particle, viz.

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

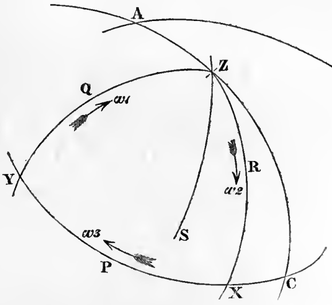
will be $\omega_3\sqrt{x^2 + y^2}$.

Now if we suppose the particle to be at a distance unity from the origin, and therefore $x^2 + y^2 = 1$, this velocity will become ω_3, and hence ω_3 is the resolved velocity in the plane of xy of any particle situated in that plane, at a distance

unity from the origin. Since the particle is rigidly connected with the origin, it is evident that this velocity takes place perpendicularly to the line joining the particle and the origin; and also since $\frac{dx}{dt}$ is negative, it is evident that this velocity tends to move the particle from the axis of x towards that of y.

In the same way it may be proved that ω_1 and ω_2 are similar velocities with respect to the planes of yz and zx, respectively; ω_1 tending from the axis of y towards that of z, and ω_2 tending from the axis of z towards that of x.

71. To give clear ideas, we shall represent the manner in which these velocities tend, by means of the following figure.



Let XY, YZ, ZX be the intersections of the planes of xy, yz, zx, respectively, at the time t, with a sphere fixed in space, unity being its radius, and the origin its center; then ω_3 will be the resolved velocity along the great circle XY of any point P situated on that great circle, and ω_1 will be the resolved velocity along the great circle YZ of any point Q situated on that great circle, and ω_2 will be the resolved velocity along the great circle ZX of any point R on that great circle; and these velocities tend in the directions represented by the arrows.

It is evident from this, that ω_1, ω_2, ω_3 are also the angular velocities of the planes of yz, zx, and xy, round the axes of x, y, and z, respectively: but it may be easily seen from the equations (B), that ω_1, ω_2, ω_3 are not the angular velocities round the axes of x, y, z, of any other points of the body, except those situated in the respective co-ordinate planes.

72. We may determine the position and motion of the body by means of these velocities, as follows.

Let A be any fixed point on the surface of the sphere described before (see the figure), and AB any fixed great circle; draw the great circle AZC to meet YX produced in C.

Let the angle $ABZ = \psi$, the angle $AZ = \theta$, and the angle $CZX = \phi$; then it is evident that these angles define completely the position of the body in space at the time t, and may be considered as the co-ordinates of the position of the body, ψ and θ being the co-ordinates of the point Z, which we shall call the pole of the body, and ϕ the additional co-ordinate requisite to determine the position of the plane of ZX, and therefore of the whole body; we may easily determine these co-ordinates from the quantities ω_1, ω_2, ω_3, as follows.

It is evident that the velocities of the point of the body coinciding with Z are,

$$\frac{d\psi}{dt}\sin\theta, \text{ perpendicular to } AZ,$$

$$\text{and } \frac{d\theta}{dt} \qquad \text{along } AZ,$$

and the velocity along CX of the point of the body coinciding with C is

$$\frac{d\psi}{dt}\sin AC, \text{ due to the variation of } \psi,$$

$$+\frac{d\phi}{dt}\sin ZC, \text{ due to the variation of } \phi;$$

i. e., since $ZC = 90^{\circ}$,

$$\frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}.$$

Now by what we have proved in (71), the velocities of the body coinciding with Z are,

$$\begin{aligned} &\omega_2 \text{ along } ZX, \\ \text{and } \quad &-\omega_1 \text{ along } ZY, \end{aligned}$$

and the velocity along CX of the point coinciding with C is ω_3; hence, since these two sets of velocities must be equivalent, we have, resolving the latter set so as to make them coincide with the former,

$$\left.\begin{aligned} \frac{d\psi}{dt}\sin\theta &= -\omega_1\cos\phi + \omega_2\sin\phi \\ \frac{d\theta}{dt} &= \omega_1\sin\phi + \omega_2\cos\phi \\ \frac{d\psi}{dt}\cos\theta + \frac{d\psi}{dt} &= \omega_3 \end{aligned}\right\} \cdots\cdots\cdots (C).$$

73. These differential equations connect ϕ, ψ, and θ with ω_1, ω_2, ω_3, and these, along with the three others (A') in (69), which connect ω_1, ω_2, ω_3 with the impressed forces, form a system of six equations connecting the six unknown quantities ω_1, ω_2, ω_3, ψ, θ, ϕ with t, and which therefore, when it is possible to do so, will enable us to solve any problem respecting the motion of a rigid body about its centre of gravity.

The equations (B) will enable us to determine the motion of any point we please of the body, should it be requisite to do so.

74. These equations will be sufficient for our present purpose, namely, the determination of the Earth's motion about its centre of gravity; we shall hereafter recur to this subject, and deduce several interesting consequences from these equations.

CHAPTER VI.

PRECESSION AND NUTATION.

75. We shall now make use of the equations deduced in the last Chapter, to determine the motion of the Earth round its center of gravity.

The forces which act on the Earth, are the attractions of the Sun, Moon, and other planetary bodies; but on account of the Earth's nearly spherical form, the motions of the Earth round its center of gravity, produced by these forces, are but small; hence, by the principle of the superposition of small motions, we may consider, separately and by itself, the effect of the attraction of each planetary body on the motion of the Earth round its center of gravity; we shall accordingly commence with the Sun's effect.

76. We shall first prove, that the attractions of the Sun on any particle of the Earth are the same very nearly as if the Sun's mass were condensed into his center of gravity.

Take the center of gravity of the Sun as origin; let $x'y'z'$ be the co-ordinates of any element $\delta m'$ of it, and let xyz be the co-ordinates of the attracted particle δm of the Earth.

Then, if V denote the sum of each element of the Sun, divided by its distance from δm, the attractions of the Sun on δm will be

$$\frac{dV}{dx}, \quad \frac{dV}{dy}, \quad \frac{dV}{dz}.$$

Now $V = \Sigma \dfrac{\delta m'}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}}$

$$= \Sigma \frac{\delta m'}{r} \left\{1 - \frac{2(xx' + yy' + zz')}{r} + \frac{r'^2}{r^2}\right\}^{-\frac{1}{2}},$$

putting r and r' for $x^2 + y^2 + z^2$, and $x'^2 + y'^2 + z'^2$, respectively

$$= \Sigma \frac{\delta m'}{r} \left\{1 + \frac{xx' + yy' + zz'}{r}\right\};$$

expanding and neglecting the squares, &c. of the very small quantities $\frac{x'}{r}$, $\frac{y'}{r}$, $\frac{z'}{r}$, and $\frac{r'}{r}$

$$= \frac{1}{r} \Sigma \delta m'.$$

Since $\Sigma \delta m x'$, $\Sigma \delta m y'$, $\Sigma \delta m z'$ are zero, because the origin is center of gravity.

Hence $V = \dfrac{m'}{r}$;

and therefore, neglecting the squares &c. of very small quantities, V, and consequently the attractions $\frac{dV}{dx}$, $\frac{dV}{dy}$, $\frac{dV}{dz}$, are the same as if the whole mass of the Sun were collected into its center of gravity.

77. Now take the principal axes of the Earth, passing through its center of gravity, as the axes of co-ordinates, the polar axis being that of z; let xyz be the co-ordinates of any element δm of the Earth, and $x'y'z'$ the co-ordinates of the Sun, supposed to be condensed into his center of gravity; then, m' being the mass of the Sun, his attractions on δm, parallel to the axes of x and y, will be

$$X = \frac{m'(x'-x)}{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{3}{2}}},$$

and $Y = \dfrac{m'(y'-y)}{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{3}{2}}}$;

and hence the quantity N, (see the equations (A') of the last Chapter) which equals $\Sigma\delta m\,(Yx - Xy)$, will become

$$N = m'\Sigma\delta m \left\{ \frac{(y' - y)\,x - (x' - x)\,y}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{3}{2}}} \cdots \right\}$$

$$= m'\Sigma\delta m\,\{y'x - x'y)\left[r'^2 - 2\,(xx' + yy' + zz') + r^2\right]^{-\frac{3}{2}}\},$$

r' and r being the distances of $\delta m'$ and δm from the origin, and r' being therefore very large compared with x, y, z, or r; hence, expanding and neglecting the squares of very small quantities, we have

$$N = m'\Sigma\delta m \left\{ (y'x - x'y) \cdot \frac{1}{r'^3} \cdot \left(1 + 3\,\frac{xx' + yy' + zz'}{r'^2}\right) \right\};$$

or, since the origin is center of gravity, and therefore $\Sigma\delta m x$, $\Sigma\delta m y$, $\Sigma\delta m z$, each zero, and since the axes are principal axes, and therefore $\Sigma\delta m xy$, $\Sigma\delta m yz$, $\Sigma\delta m zx$, each zero, we have

$$N = \frac{3m'x'y'}{r'^2}\,\Sigma\delta m\,(x^2 - y^2)$$

$$= \frac{3m'x'y'}{x'^5}\,(B - A),$$

B and A being the same as in the last Chapter.

Similarly,

$$M = \frac{3m'z'x'}{r'^5}\,(A - C),$$

$$L = \frac{3m'y'z'}{r'^5}\,(C - B).$$

Hence the equations (A'), in the last Chapter, become

$$A\,\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 = \frac{3m'y'z'}{r'^5}\,(C - B),$$

$$B\,\frac{d\omega_2}{dt} + (A - C)\,\omega_3\omega_1 = \frac{3m'z'x'}{r'^5}\,(A - C),$$

$$C\,\frac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 = \frac{3m'x'y'}{r'^5}\,(B - A).$$

78. These equations simplify very much in the case of the Earth, for the polar axis being that of z, the moments of inertia round all axes in the plain of xy will be the same, since the Earth is symmetrical with respect to the polar axis; hence $B = A$, and the last equation will become

$$\frac{d\omega_3}{dt} = 0,$$

and therefore $\omega_3 =$ constant $= n$ suppose.

That is to say, by (71), the angular velocity of the plane of the equator round the fixed axis with which the polar axis coincides at any time, is a constant quantity.

Now if n' be the Sun's mean angular velocity relative to the Earth,

$$n'^2 = \frac{m + m'}{r'^3}, \quad m \text{ being the Earth's mass,}$$

$$= \frac{m'}{r'^3}, \text{ very nearly, } m \text{ being very small com-}$$

pared with m';

hence, and putting β for $\frac{C - A}{A}$, the other two equations become

$$\frac{d\omega_1}{dt} + n\beta . \omega_2 = 3n'^2 . \frac{y'z'}{r'^2}\beta,$$

$$\frac{d\omega_2}{dt} - n\beta\omega_1 = -3n'^2\frac{z'x'}{r'^2}\beta.$$

79. Now in the figure (page 66), let S be the point where a line drawn from the center of the Earth to the Sun meets the fixed sphere mentioned in page 66; draw the great circle SZ, then SZ will be the Sun's north polar distance, which we shall denote by Δ, and SZX will be the Sun's hour-angle relative to the meridian plane ZX, which we shall denote by h: it is evident that Δ and h are the polar co-ordinates of the Sun, and therefore we have

$$x' = r' \sin \Delta \cos h,$$
$$y' = r' \sin \Delta \sin h,$$
$$z' = r' \cos \Delta.$$

hence $$\frac{y'z'}{r'^2} = \sin \Delta \cos \Delta \sin h,$$

$$\frac{z'x'}{r'^2} = \sin \Delta \cos \Delta \cos h;$$

and hence our equations become

$$\frac{d\omega_1}{dt} + n\beta\omega_2 = 3n'^2\beta \sin \Delta \cos \Delta \sin h,$$

$$\frac{d\omega_2}{dt} - n\beta\omega_1 = -3n'^2\beta \sin \Delta \cos \Delta \cos h.$$

We shall take a year as the unit of time, and hence n', which $= \dfrac{2\pi}{\text{a year}}$, will become 2π, and n will be about 365, also we shall shew that β is about $= \dfrac{1}{330}$.

Now h varies in consequence of the Earth's diurnal rotation, and also in consequence of the Sun's motion and the motion of the polar axis; but the part of its variation due to the former cause is very much greater than that due to the latter causes, hence we may put

$$\frac{dh}{dt} = -n + \delta n;$$

where δn is a small quantity compared with n, depending on the motions of the Sun and polar axis, we put $-n$ because h decreases with the time.

Hence in the first of our equations, putting

$$\frac{d\omega_1}{dt} = \frac{d\omega_1}{dh}\frac{dh}{dt} = \frac{d\omega_1}{dh}(-n + \delta n),$$

it becomes

$$\frac{d\omega_1}{dh} + \frac{n\beta}{-n + \delta n}\omega_2 = \frac{3n'^2\beta}{-n + \delta n} \sin \Delta \cos \Delta \sin h;$$

or, neglecting δn compared with n,

$$\frac{d\omega_1}{dh} - \beta\omega_2 = -3n'^2 \frac{\beta}{n} \sin \Delta \cos \Delta \sin h.$$

Now the second member of this equation is very much smaller than the first, on account of being multiplied by $\frac{1}{n}$, hence in it we may, when integrating, consider the periodical quantity Δ as invariable, since it varies very slowly compared with h; and therefore putting, for brevity,

$$\frac{3n'^2\beta}{n} \sin \Delta \cos \Delta = \gamma,$$

our equation becomes

$$\frac{d\omega_1}{dh} - \beta\omega_2 = -\gamma \sin h \ldots (1),$$

where γ is a very small quantity which may be considered invariable in integrating.

In the same way we shall have

$$\frac{d\omega_2}{dh} + \beta\omega_1 = \gamma \cos h \ldots (2).$$

80. Now differentiating (1), and adding (2) multiplied by β to it, we have

$$\frac{d^2\omega_1}{dh^2} + \beta^2\omega_1 = (-\gamma + \gamma\beta) \cos h$$

$$= -\gamma \cos h,$$

neglecting β compared with unity.

The integral of this equation will be

$$\omega_1 = A \cos(\beta h + B) + C \cos h,$$

where A and B are arbitrary constants, and C a constant to be determined by substitution. Now A and B, since they depend only on the initial circumstances of the motion, are independent of the Sun's action; and, as it is our object to

determine the effect of that action alone, we shall omit the term

$$A \cos(\beta h + B),$$

which does not depend on it, and we shall have simply

$$\omega_1 = C \cos h,$$

so far as the Sun's action is concerned; substituting this value in the equation, we find

$$C(-1 + \beta^2) = -\gamma,$$

or $C = \gamma$, neglecting β^2; hence

$$\omega_1 = \gamma \cos h.$$

In like manner we shall have, differentiating the equation (2), and subtracting (1) multiplied by β from it,

$$\frac{d^2\omega_2}{dh} + \beta^2\omega_2 = -\gamma \sin h;$$

and therefore, as before,

$$\omega_2 = \gamma \sin h.$$

81. Having thus determined ω_1 and ω_2, we shall substitute their values in the equations (C) (last Chapter), in order to determine θ and ϕ, and so find the position and motion of the pole; we have then, substituting the values of ω_1 and ω_2 just obtained in the two first of the equations (C),

$$\frac{d\psi}{dt}\sin\theta = \gamma \sin h \sin\phi - \gamma \cos h \cos\phi$$

$$= -\gamma \cos(h + \phi),$$

$$\text{and } \frac{d\theta}{dt} = \gamma \sin h \cos\phi + \gamma \cos h \sin\phi$$

$$= \gamma \sin(h + \phi).$$

Now in the figure (page 66), the angle SZX is h, and the angle XZC is ϕ, hence the angle SZC is $\phi + h$; therefore if we take the point A to be the pole of the ecliptic (which we may do since its position is arbitrary), it is evident that $\phi + h$ or SZC will be the Sun's right ascension

-90°; hence, if a be the Sun's right ascension, we shall have

$$\phi + h = a - 90^\circ,$$

and hence our equations become

$$\frac{d\psi}{dt}\sin\theta = -\gamma \sin a,$$

$$\frac{d\theta}{dt} = -\gamma \cos a;$$

or, putting for γ its value,

$$\frac{d\psi}{dt}\sin\theta = -\frac{3n'^2\beta}{n}\sin\Delta\cos\Delta\sin a,$$

$$\frac{d\theta}{dt} = -\frac{3n'^2\beta}{n}\sin\Delta\cos\Delta\cos a.$$

82. Now if l be the Sun's longitude, l, a, and $90^\circ - \Delta$ are the sides of a right-angled spherical triangle, the right angle being opposite l, and θ (the obliquity of the ecliptic) being the angle opposite $90^\circ - \Delta$; hence since, by Napier's rules,

$$\cos\Delta = \sin\theta\sin l \;\ldots\ldots\ldots\ldots\; (1),$$

$$\cos l = \sin\Delta\cos a \;\ldots\ldots\ldots\; (2),$$

$$\text{and } \sin a = \cot\Delta\cot\theta \;\ldots\ldots\ldots\ldots\; (3),$$

we have

$$\sin\Delta\cos\Delta\sin a = \cos^2\Delta\cot\theta, \text{ by } (3),$$

$$= \sin\theta\cos\theta\sin^2 l, \text{ by } (1),$$

$$\text{and } \sin\Delta\cos\Delta\cos a = \cos l\cos\Delta, \text{ by } (2),$$

$$= \sin\theta\sin l\cos l, \text{ by } (1);$$

hence our equations become

$$\frac{d\psi}{dt} = -\frac{3n'^2\beta}{n}\cos\theta\sin^2 l,$$

$$\frac{d\theta}{dt} = -\frac{3n'^2\beta}{n}\sin\theta\cos l.$$

83. Now $\frac{\beta}{n}$ being very small, we may in integrating these equations consider θ invariable in the terms multiplied by $\frac{\beta}{n}$; and we may also, for the same reason, consider l to vary uniformly, and therefore put $\frac{dl}{dt} = n'$; hence, putting

$$\frac{d\psi}{dt} = \frac{d\psi}{dl}\,\frac{dl}{dt} = \frac{d\psi}{dl}\,n',$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dl}\,n',$$

and $\sin^2 l = \frac{1}{2}(1 - \cos 2l)$,

our equations become

$$\frac{d\psi}{dl} = -\frac{3}{2}\,\frac{n'}{n}\,\beta \cos\theta\,(1 - \cos 2l),$$

$$\frac{d\theta}{dl} = -\,3\,\frac{n'}{n}\,\beta \sin\theta \cos l;$$

hence, integrating

$$\psi + C = -\frac{3}{2}\,\frac{n'}{n}\,\beta \,.\, \cos\theta\,(l - \tfrac{1}{2}\sin 2l),$$

$$\theta + C' = 3\,\frac{n'}{n}\,\beta \sin\theta \sin l.$$

In the second members of these equations θ may be considered as the mean obliquity of the ecliptic, and may therefore be determined by astronomical observations; and $\frac{n'}{n}$, the ratio of a year to a day, may be similarly determined; and thus we may put our equations in the form

$$\psi + C = -\,e'\beta l + f'\beta \sin 2l,$$

$$\theta + C = g'\beta \sin l,$$

where e', f', g', are numerical quantities got from observation.

These equations determine the effect of the Sun's attraction on the motion of the Earth round its center of gravity.

84. To determine the effect of the Moon on the Earth's motion round its center of gravity, we may proceed in exactly a similar way, merely supposing the symbols which before referred to the Sun now refer to the Moon, and altering them accordingly, as follows:

Let $m_{,}$, $n_{,}$, and $r_{,}$ be the mass, mean angular velocity, and distance of the Moon, then as in Art. (78), we shall have

$$n_{,}^{2} = \frac{m + m_{,}}{r_{,}^{3}},$$

here m instead of being much smaller than $m_{,}$ as before, is much larger than it, and therefore cannot be neglected as before,

$$\text{let } \lambda = \frac{m}{m_{,}}, \text{ then}$$

$$n_{,}^{2} = \frac{\lambda m_{,} + m_{,}}{r_{,}^{3}},$$

$$\text{and therefore } \frac{m_{,}}{r_{,}^{3}} = \frac{n_{,}^{2}}{\lambda + 1};$$

hence, in changing our equations so as to refer to the Moon, we must in substituting for n'^{2} put $\dfrac{n_{,}^{2}}{\lambda + 1}$ instead of $n_{,}^{2}$ for it.

Hence, if we put dashes under the letters to denote that they refer to the Moon, we shall have for the Moon's effect on the Earth's motion round its center of gravity,

$$\psi_{,} + C_{,} = -\frac{3n_{,}^{2}\beta}{2n(\lambda+1)} \cos\theta_{,} \left(t - \frac{1}{2n_{,}} \sin 2n_{,}t\right),$$

$$\theta_{,} + C_{,}' = \frac{3n_{,}\beta}{2n(\lambda+1)} \sin\theta_{,} \sin n_{,}t,$$

Of course the point A in the figure, (page 66), is now supposed to be the pole of the Moon's orbit, and not the pole of the ecliptic.

85. In these expressions the coefficients are much smaller than before; so small that the periodical quantities multiplied by them, viz. sin $2n_,t$ and sin $n_,t$, which go through all their values in half a month, and a month respectively, may be neglected, this will give

$$\psi_, + C_, = -\frac{3n_,^2\beta}{2n(\lambda+1)}\cos\theta_, . t,$$

and $\theta_, + C'_, = 0$;

$$\text{and therefore } \frac{d\psi_,}{dt} = -\frac{3n_,^2\beta}{2n(\lambda+1)}\cos\theta_,,$$

$$\text{and } \frac{d\theta_,}{dt} = 0,$$

Which equations prove, that the effect of the Moon's attraction (omitting very small periodical quantities of short period) is to produce a motion of the pole of the Earth perpendicular to the great circle AZ, i.e., the great circle joining the pole of the Earth and the pole of the Moon's orbit; and the velocity with which this motion takes place

$$\left(\text{i.e. } \frac{d\psi_,}{dt}\sin\theta_,\right), \text{ is}$$

$$-\frac{3n_,^2\beta}{2n(\lambda+1)}\cos\theta_,\sin\theta_,.$$

We shall resolve this velocity along and perpendicular to the great circle, joining the pole of the Earth and the pole of the ecliptic, in order to get our quantities measured in the same way as before in the case of the Sun, and so determine the variations (due to the Moon's action), of the angles ψ and θ, which refer to the pole of the ecliptic.

86. Let ι be the inclination of the Moon's orbit to the plane of the ecliptic; then it is evident that ι, θ, and $\theta_,$ form the sides of a spherical triangle.

Let σ be the angle made by θ and $\theta_{,}$, then resolving the velocity

$$\frac{3n_{,}^{2}\beta}{2n(\lambda+1)}\cos\theta_{,}\sin\theta_{,},$$

(which acts perpendicularly to $\theta_{,}$), along and perpendicular to θ we find for the resolved parts,

$$-\frac{3n_{,}^{2}\beta}{2n(\lambda+1)}\cos\theta_{,}\sin\theta_{,}\cos\sigma \quad \text{perpendicular to } \theta,$$

$$\text{and } -\frac{3n_{,}^{2}\beta}{2n(\lambda+1)}\cos\theta_{,}\sin\theta_{,}\sin\sigma \quad \text{along } \theta,$$

and hence, since $\frac{d\psi}{dt}\sin\theta$ and $\frac{d\theta}{dt}$ are these velocities, we have the following equations to determine the effect of the Moon's action on ψ and θ, viz.

$$\frac{d\psi}{dt} = -\frac{3n_{,}^{2}\beta}{2n(\lambda+1)}\,\frac{\cos\theta_{,}\sin\theta_{,}\cos\sigma}{\sin\theta},$$

$$\frac{d\theta}{dt} = -\frac{3n_{,}^{2}\beta}{2n(\lambda+1)}\cos\theta_{,}\sin\theta_{,}\sin\sigma.$$

Now in the triangle of which ι, θ, and $\theta_{,}$ are the sides, σ is the angle opposite ι, and if ☊ be the longitude of the Moon's node, it is evident that ☊ is the angle opposite $\theta_{,}$; hence we have

$$\cos\iota = \cos\theta\cos\theta_{,} + \sin\theta\sin\theta_{,}\cos\sigma, \;\ldots\; (1)$$

$$\cos\theta_{,} = \cos\iota\cos\theta + \sin\iota\sin\theta\cos☊, \;\ldots\; (2)$$

$$\frac{\sin\theta_{,}}{\sin ☊} = \frac{\sin\iota}{\sin\sigma}; \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3)$$

hence $\dfrac{\cos\theta_{,}\sin\theta_{,}\cos\sigma}{\sin\theta}$

$$= \cos\theta_{,}\cdot\frac{\cos\iota - \cos\theta\cos\theta_{,}}{\sin^{2}\theta}, \quad \text{by (1)}.$$

$$= (\cos\iota\cos\theta + \sin\iota\sin\theta\cos\Omega).\frac{\cos\iota\sin^2\theta - \sin\iota\cos\theta\sin\theta\cos\Omega}{\sin^2\theta}$$

$$= \cos^2\iota\cos\theta + \sin\iota\cos\iota\frac{(\sin^2\theta - \cos^2\theta)}{\sin\theta}\cos\Omega - \sin^2\iota\cos\theta\cos^2\Omega$$

$$= \cos\theta(\cos^2\iota - \tfrac{1}{2}\sin^2\iota) - \tfrac{1}{2}\sin 2\iota\frac{\cos 2\theta}{\sin\theta}\cos\Omega - \tfrac{1}{2}\sin^2\iota\cos\theta\cos 2\Omega;$$

and also

$$\cos\theta_{,}\sin\theta_{,}\sin\sigma$$

$$= (\cos\iota\cos\theta + \sin\iota\sin\theta\cos\Omega)\sin\iota\sin\Omega$$

$$= \tfrac{1}{2}\sin 2\iota\cos\theta\sin\Omega + \tfrac{1}{2}\sin^2\iota\sin\theta\sin 2\Omega.$$

In these expressions Ω alone may be considered as variable, all the other quantities varying very slowly, and within very small limits; also, since these expressions are to be multiplied by a very small coefficient, we may, in the periodical terms, neglect $\sin^2\iota$, since ι is not much more than 5^0.

Also, if ν be the mean angular retrograde velocity of the Moon's nodes, we may put

$$\frac{d\Omega}{dt} = -\nu;$$

and hence we shall have as in former cases

$$\frac{d\psi}{d\Omega} = \frac{3n_{,}^2\beta}{2n\nu(\lambda+1)}\cos\theta\left\{\cos^2\iota - \frac{\sin^2\iota}{2} - \sin 2\iota\cot 2\theta\cos\Omega\right\},$$

$$\frac{d\theta}{d\Omega} = \frac{3n_{,}^2\beta}{4n\nu(\lambda+1)}\cos\theta\sin 2\iota\sin\Omega;$$

and hence, integrating,

$$\psi + C = \frac{3n_{,}^2\beta}{2n\nu(\lambda+1)}\cos\theta\left\{\left(\cos^2\iota - \frac{\sin^2\iota}{2}\right)\Omega - \sin 2\iota\cot 2\theta\sin\Omega\right\},$$

$$\theta + C' = -\frac{3n_{,}^2\beta}{4n\nu(\lambda+1)}\cos\theta\sin 2\iota\cos\Omega;$$

or, as before, we may put these equations in the form

$$\psi + C = e_{,} \frac{\beta}{\lambda + 1} \Omega - f_{,} \frac{\beta}{\lambda + 1} \sin \Omega ,$$

$$\theta + C' = - g_{,} \frac{\beta}{\lambda + 1} \cos \Omega .$$

When $e_{,}$, $f_{,}$, and $g_{,}$ are numerical quantities, got by observation.

Thus we have determined the effect of the Moon's attraction on the motion of the Earth, round its center of gravity.

87. The effects of the other planetary bodies are very small indeed, and we shall neglect them; hence, adding together the effects of the Sun and Moon, we find for the whole motion of the Earth round its center of gravity, so far as it is affected by external attractions,

$$\psi + C = - \beta \left(e' l - \frac{e_{,}}{\lambda + 1} \Omega\right) + f' \beta \sin 2 l$$
$$- f_{,} \frac{\beta}{\lambda + 1} \sin \Omega ,$$

$$\theta + C' = g' \beta \sin l - g_{,} \frac{\beta}{\lambda + 1} \cos \Omega .$$

The first term of the expression for $\psi + C$ is non-periodical, and its rate of variation is

$$- \beta \left(e' \frac{dl}{dt} - \frac{e'}{\lambda + 1} \frac{d\Omega}{dt}\right),$$

$$\text{or} \quad - \beta \left(e' n' + \frac{e_{,} \nu}{\lambda + 1}\right);$$

which being constant and negative represents a uniform retrograde motion of the pole in longitude: it gives rise to what is called the *precession of the equinoxes*, because, in consequence of it, the first point of Aries moves constantly backwards, and therefore the equinox occurs sooner than it otherwise would every year.

Observation shews that this retrograde motion of the pole in longitude is about 50″.1 per year; hence we ought to have

$$50''.1 = \beta\left(e'n' + \frac{e_{\prime}\nu}{\lambda + 1}\right) \ldots\ldots\ldots (1).$$

The other terms of $\psi + C$ and $\theta + C'$ are periodical, depending on the longitude of the Sun and of the Moon's nodes; they are called the solar and lunar nutations. Observation shews that the coefficient of sin ☊ is about 18″, and that of cos ☊ about 9″.5, the coefficients of the other terms are much smaller; hence we ought to have

$$18'' = f_{\prime}\frac{\beta}{\lambda + 1} \ldots\ldots\ldots\ldots (2),$$

$$\text{and } 9''.6 = g_{\prime}\frac{\beta}{\lambda + 1} \ldots\ldots\ldots\ldots (3).$$

88. Since $e_{\prime}$, $f_{\prime}$, and $g_{\prime}$ are known numerical quantities, it is evident, that from (1) combined with (2) or (3), we may eliminate β and find λ; the result is

$$\lambda = \text{about } 70,$$

thus by observation on precession and nutation we may determine λ, which is the ratio of the Earth's mass to that of the Moon.

By the same equations we may determine β, the result is

$$\beta = \text{about } .00319, \text{ or } \frac{1}{330}.$$

89. Now β may be also calculated by integration, if we know the law of arrangement of the Earth's mass; we shall calculate β, assuming the results arrived at in Chap. III, and if we find that the value of β thus obtained coincides with that just determined by observation, it is evident that we shall have an additional proof of the correctness of our hypotheses in Chap. III.

90. We have evidently

$$\beta = \frac{\Sigma \delta m\,(x^2 - z^2)}{\Sigma \delta m\,(y^2 + z^2)} = \frac{\Sigma \delta m\,(x^2 - z^2)}{\Sigma \delta m\,(x^2 + z^2)}, \text{ since } A = B;$$

x, y, z being the co-ordinates of any particle (δm) of the Earth, the polar axis being the axis of z; or using the polar co-ordinates as before,

$$\beta = \frac{\int_0^{2\pi}\int_{-1}^{1}\int_0^{r_1} \rho r^4 \{(1 - \mu^2)\cos^2\phi - \mu^2\}\, dr\, d\mu\, d\phi}{\int_0^{2\pi}\int_{-1}^{1}\int_0^{r_1} \rho r^4 \{(1 - \mu^2)\cos^2\phi + \mu^2\}\, dr\, d\mu\, d\phi}.$$

Now putting $\frac{1}{2} + \frac{1}{2}\cos 2\phi$ for $\cos^2\phi$, and integrating relatively to ϕ, observing that r does not contain ϕ, the numerator of β becomes

$$2\pi \int_{-1}^{1}\int_0^{r_1} \rho r^4 \left(\frac{1}{2} - \frac{3\mu^2}{2}\right) dr\, d\mu$$

$$= \frac{3\pi}{5}\int_{-1}^{1}\int_0^{a_1} \rho \frac{d\,(r^5)}{da}\left(\tfrac{1}{3} - \mu^2\right) da\, d\mu,$$

which, putting

$$r^5 = a^5\left\{1 + 5\epsilon\left(\tfrac{1}{3} - \mu^2\right)\right\},$$

and observing the property of Laplace's coefficients in Art. 21, becomes

$$3\pi \int_0^a \rho \frac{d\,(a^5\epsilon)}{da}\int_{-1}^{1}\left(\tfrac{1}{3} - \mu^2\right)^2 d\mu\, da.$$

Now $\int_{-1}^{1}\left(\tfrac{1}{3} - \mu^2\right)^2 d\mu = \int_{-1}^{1}\left(\frac{1}{9} - \frac{2}{3}\mu^2 + \mu^4\right) d\mu,$

$$= 2\left(\frac{1}{9} - \frac{2}{9} + \frac{1}{5}\right),$$

$$= \frac{8}{45};$$

hence our integral becomes

$$\frac{8\pi}{15}\int_0^a \rho \frac{d\,(a^5\epsilon)}{da}\, da.$$

In like manner the denominator of β becomes

$$2\pi \int_{-1}^{1} \int_{0}^{r_1} \rho r^4 \left(\frac{1}{2} + \frac{\mu^2}{2}\right) dr\, d\mu$$

$$= \frac{\pi}{5} \int_{-1}^{1} \int_{0}^{a_1} \rho \frac{d(r^5)}{da} \left\{\frac{4}{3} - \left(\frac{1}{3} - \mu^2\right)\right\} da\, d\mu$$

$$= \frac{\pi}{5} \int_{-1}^{1} \int_{0}^{a_1} \rho \frac{d(a^5)}{da} \left\{\frac{4}{3} - \left(\frac{1}{3} - \mu^2\right)\right\} da\, d\mu.$$

Since, on account of the smallness of β, we may neglect ϵ in its denominator.

Hence, observing the property in Art. 21, the denominator of β becomes

$$\frac{8\pi}{15} \int_{0}^{a_1} \rho \frac{d(a^5)}{da}\, da \quad \text{which} = \frac{8\pi}{3} \int_{0}^{a_1} \rho a^4\, da,$$

hence we have

$$\beta = \frac{1}{5} \cdot \frac{\int_{0}^{a_1} \rho \frac{d(a^5 \epsilon)}{da}\, da}{\int_{0}^{a_1} \rho a^4\, da}.$$

Now the equation for determining ϵ in Art. 53, gives

$$-\frac{\epsilon_{\prime}}{a_{\prime}} \int_{0}^{a_1} \rho a^2 da + \frac{1}{5 a_{\prime}^3} \int_{0}^{a_1} \rho \frac{d(a^5 \epsilon)}{da}\, da + \frac{\omega^2 a_{\prime}^2}{8\pi} = 0,$$

$$\text{hence} \quad \beta = \frac{a_{\prime}^2 \epsilon_{\prime} \int_{0}^{a_1} \rho a^2 da - \frac{\omega^2 a_{\prime}^5}{8\pi}}{\int_{0}^{a_1} \rho a^4\, da}$$

$$= \left(\epsilon_{\prime} - \frac{m}{2}\right) \frac{a_{\prime}^2 \int_{0}^{a_1} \rho a^2 da}{\int_{0}^{a_1} \rho a^4\, da};$$

$$\text{putting} \quad \frac{\omega^2 a_{\prime}^{3}}{4\pi \int_0^{a_1} \rho a^2 \, da} = m;$$

m being, as before, the ratio of the centrifugal force to gravity at the equator. We can go no farther in calculating β without knowing the law of density; hence, taking the law already assumed, and substituting in the integrals and performing the integrations, we shall find

$$\beta = \text{about } .0031359, \text{ or } \frac{1}{330}.$$

91. Hence this value of β coincides with that got from observation, in Art. 88, and we have therefore an additional proof of the hypothesis of the Earth's fluidity, or rather, of the assumed law of density; for since this result cannot be obtained without assuming the law of density, it is not of much value in proving the hypothesis of original fluidity; but we may consider that hypothesis as well established by previous results, and then the coincidence of the values of β will go to proving the probability of the assumed law of density.

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